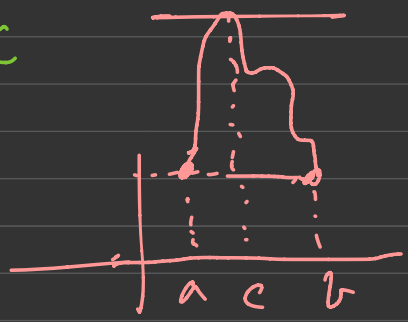
Mean Value Theorem Discussion

www.fractionshub.com

Arka Sinha

Mean Value Theorem



1) Rolle's MVT.

$f : \mathbb{R} \to \mathbb{R}$, diff'ble,

$f(a) = f(b) \Rightarrow \exists\, c \in (a, b)$ s.t.

$f'(c) = 0$

Qn $f : \mathbb{R} \to \mathbb{R}$, $f\left(\frac{1}{n}\right) = 0$

$\forall\, n \in \mathbb{N}$, f is twice diff'ble.

Prove $f''(0) = 0$.

Soln: $\underbrace{f\left(\frac{1}{n}\right)}_{=0} \longrightarrow f(0)$

$\therefore f(0) = 0.$

f diff'ble,

$$\therefore f'(0) = \lim_{n\to\infty} \frac{f(1/n) - f(0)}{1/n}$$

$$= 0$$

Since $f\left(\frac{1}{n}\right) = f\left(\frac{1}{n+1}\right) = 0.$

$\therefore \forall n, \exists c_n \in \left(\frac{1}{n+1}, \frac{1}{n}\right)$, s.t.

$f'(c_n) = 0$ (Rolles MVT)

$$\frac{1}{n+1} < c_n < \frac{1}{n} \Rightarrow c_n \to 0$$

(Sandwich!)

f is twice diff'ble,

$$\therefore f''(0) = \lim_{n\to\infty} \frac{f'(c_n) - f'(0)}{c_n}$$

$$\Rightarrow f''(0) = 0$$

$\blacksquare$

<u>Qn</u> $f : [0,1] \to \mathbb{R}$

$f(0) = 0$ & $|f'(x)| \leq |f(x)| \ \forall x \in (0,1)$

Show that $f(x) = 0 \ \forall x \in [0,1]$

Soln: Pick any $x_0 \in (0,1]$

$$\therefore \frac{f(x_0) - 0}{x_0} = f'(c_1) \quad c_1 \in (0, x_0)$$

$$\therefore f(x_0) = f'(c_1)\, x_0$$

$$\Rightarrow |f(x_0)| = |f'(c_1)|\, x_0 \leq |f(c_1)|\, x_0$$

$$\therefore |f(x_0)| \leq |f(c_1)|\, x_0$$

Similarly, $|f(c_1)| \leq |f(c_2)|\, c_1$.

$$\therefore |f(x_0)| \leq |f(c_k)|\, x_0\, c_1 c_2 \ldots c_{k-1}$$

f is bounded in $[0,1]$.

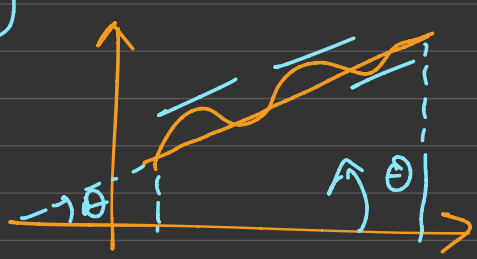
$\therefore |f(x)| \leq B x_0 c_1^{k-1} \quad \forall k \in \mathbb{N}.$

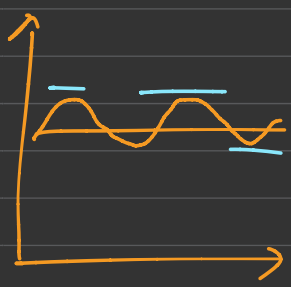
$k \to \infty \quad , \quad c_1^{k-1} \longrightarrow 0$

$\therefore |f(x)| \leq 0 \iff f(x) = 0.$

$\blacksquare$

Lagrange Mean Value Theorem:
(LMVT)





$f : \mathbb{R} \to \mathbb{R}$, diffble. $\forall a, b \in \mathbb{R}$.

$\forall a, b \in \mathbb{R}$, $\exists c \in (a, b) \ni$

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

Imp

Qu $f : \mathbb{R} \to \mathbb{R}$, $f''(x) > 0$.

Prove that $f(x + f'(x)) \geq f(x)$.

Soln Fix $x \in \mathbb{R}$.

If $f'(x) = 0$, done.

If $f'(x) < 0$, By LMVT,

then, $\exists\, c \in [f'(x)+x,\ x]$

s.t. $f(x) - f(f'(x)+x) = -f'(x)f'(c)$

$f'' > 0$, $\Rightarrow f' \uparrow$.

$\therefore f'(c) < f'(x) < 0$

$\therefore f(x) - f(f'(x)+x) < 0$

Similarly for $f'(x) > 0$

$\blacksquare$

Qu $f:[a,b] \to \mathbb{R}$ is diff'ble on (a,b).

Show, $\exists\, c \in (a,b) \ni b-a \geq \pi$

$$f'(c) < 1+f^2(c)$$

Proof: Assume that $f'(c) \geq 1+f^2(c)$ $\forall\, c \in (a,b)$

$$\therefore \frac{f'(c)}{1+f^2(c)} \geq 1 \longrightarrow \left(\tan^{-1} f(x)\right)'$$

Take $g(x) = \tan^{-1} f(x)$

By LMVT,

$$\frac{g(b)-g(a)}{b-a} = g'(c) \geq 1$$

$$\therefore g(b)-g(a) \geq b-a \geq \pi$$

but $g(b)-g(a) = \tan^{-1} f(b) - \tan^{-1} f(a)$

$$< \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi$$

$(\Rightarrow\Leftarrow)$

$\blacksquare$

H.W

1) $f : [1,2] \longrightarrow \mathbb{R}$, continuous.

$$f(x) = \sum_{n=1}^{\infty} \frac{f(x^{1/n})}{2^n}$$

Prove that f is constant.

2) $f : [0,5] \longrightarrow \mathbb{R}$ is differentiable.
(At the end points, assume the one sided derivatives exist).

If $\nexists\, x \in [0,5] \ni f(x) = 0 = f'(x)$
Then show that $S = \{x \mid f(x) = 0\}$ is finite.