# Ramsey Theory Discussion

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# Introduction to Ramsey Theory

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# ✻ An Interesting Result

We start with the following well-known result: *"Among 6 people, there are 3, all of whom know each other or there are 3, none of whom know each other."*

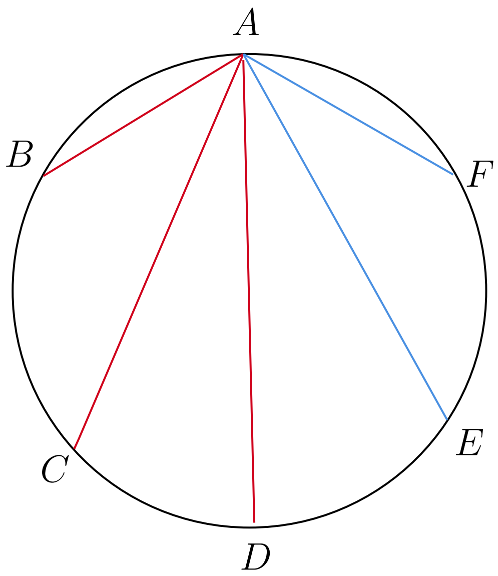
***Proof:*** Consider 6 points representing 6 persons on a circle. We draw a red line between any two persons who know each other and draw a blue line between any two persons who do not know each other. We have to show that there exists a blue triangle or a red triangle.

We first pick up any vertex and draw the red and blue lines from that vertex to the other vertices. Now, two cases may arise:

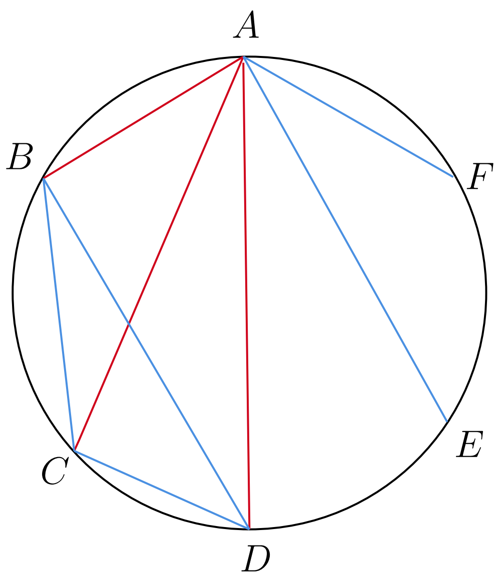
(1) The number of red lines is 3 or more

(2) The number of blue lines is 3 or more

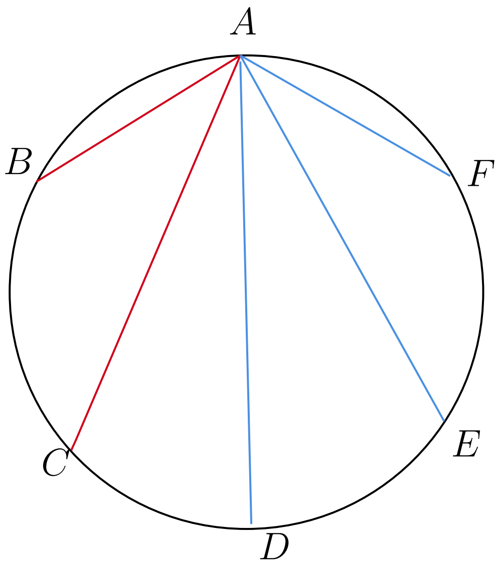
*Case 1:* This case is depicted in the figure below:




In this case, any of BC, BD and CD being red would imply that ABC,ABD or ACD is a red triangle. On the other hand, if none of these are red, then all of them are blue as shown below. In that case BCD is a blue triangle.




*Case 2:* this is depicted in the figure below:



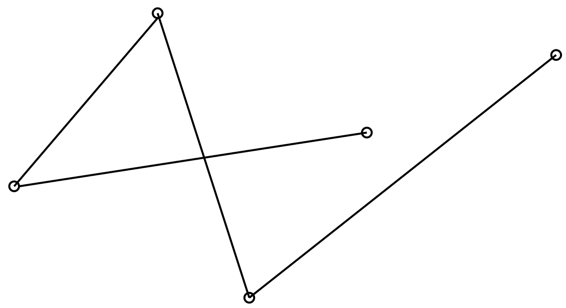

The proof in this case uses the same argument as that used in the previous part. ■

# ✻ The Ramsey Theorem

Before we go into the **Ramsey Theorem**, we define certain things about a graph.

A **graph** $G$ is a triple consisting of a vertex set $V(G)$, an edge set $E(G)$ and a relation that associates with each edge two vertices called its endpoints.

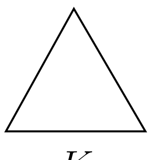
If, for any edge, its endpoints are distinct and no two edges share the same endpoints, then the graph is called simple.



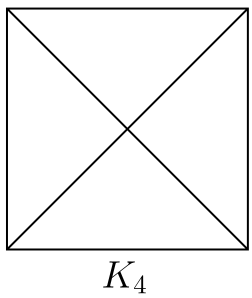
The **degree** of any vertex is the number of edges incident at that vertex.

Check yourself that the sum of degrees of all the vertices is twice the number of edges i.e. $\sum_{v \in V(G)} d_v = 2|E(G)|$.

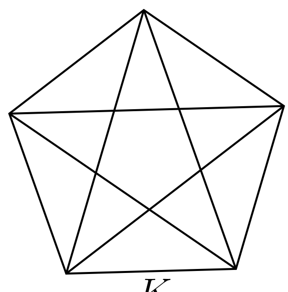
A **clique** is a a set of vertices such that every pair of vertices has an edge between them. The order of a clique is its number of vertices. A clique of order $n$ is denoted by $K_n$.



$K_3$



$K_4$



$K_5$

An **independent set** is a set of vertices in such that no two of them have an edge joining them.

The **Ramsey Number** $R(s,t)$ is defined as the minimum value $n$ such that a graph of $n$ vertices contains either a clique of size $s$ or an independent set of size $t$.

It is not hard to see that $R(s,t) = R(t,s)$.

**Ramsey Theorem:** For any $s, t \geq 1$, $\exists\ R(s,t) < \infty$ such that any graph on $R(s,t)$ vertices contains either a clique of size $s$ or an independent set of size $t$ or vice-versa.

***Proof:*** _Claim:_ $R(s,t) \leq R(s,t-1) + R(s-1,t)$.

To prove the above claim, we need to show that a graph on $n = R(s,t-1)+R(s-1,t)$ vertices contains a clique of size $s$ or an independent set of size $t$ or vice-versa. We fix any vertex $v$. Then note that $v$ either has at least $R(s-1,t)$ neighbours or it has at least $R(s,t-1)$ non-neighbours.

_Case 1:_ $v$ along with *clique* of size $s-1$ forms a clique of size $s$. If there isn't a clique of size $s$, then there must be an independent set of size $t$.

_Case 2:_ $v$ along with an independent set of size $t-1$ forms an independent set of size $t$. If there isn't an independent set of size $t-1$, then there must be a clique of size $s$. This proves the claim.

Now, if $R(s,t-1)$ and $R(s-1,t)$ exist finitely, then $R(s,t)$ must also exist finitely. So, we can say by induction that for every $s, t \geq 1$, $\exists\ R(s,t) < \infty$. ■