Continuity Discussion

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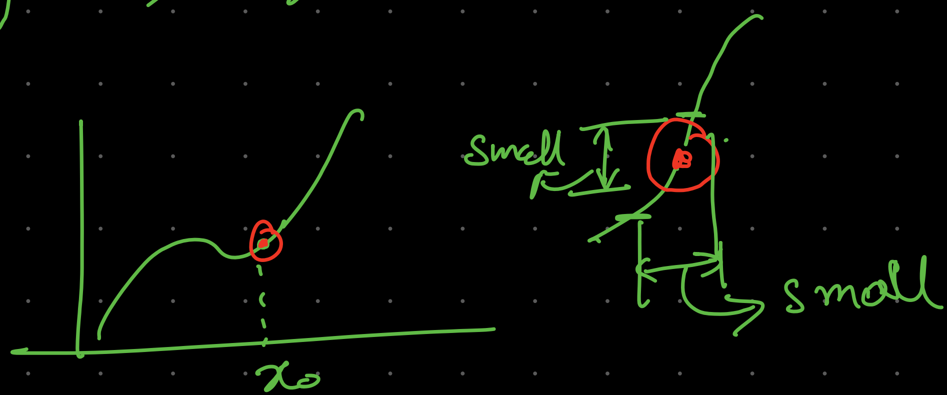
Continuity

What does it mean for a fn to be continuous at a point.



If x is near x_0,
$f(x)$ is near x_0

As x goes closer to x_0,
$f(x)$ also gets closer to $f(x_0)$

Formal Definition:

$f : S \to \mathbb{R}$ $(S \subseteq \mathbb{R})$

f is said to be continuous at x_0,
if $\forall\, \varepsilon > 0$, $\exists\, \delta > 0$, such that,

$$|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon.$$

Qn Prove that $\sin x$ is continuous at any $x_0 \in \mathbb{R}$.

Soln: Fix an x_0.

$$|\sin x - \sin x_0|$$

$$= \left|2 \sin\left(\frac{x - x_0}{2}\right) \cos\left(\frac{x_0 + x}{2}\right)\right|$$

$$\leq 2 \left| \sin \left(\frac{x - x_0}{2} \right) \right|$$

$$\leq 2 \times \frac{|x - x_0|}{2} = |x - x_0|$$

So, if I want $|\sin x_0 - \sin x| < \varepsilon$

Just take $\delta = \varepsilon$, and we are done. $\blacksquare$

<u>Defn</u>: $f : S \to \mathbb{R}$ is said to be continuous if f is continuous at all $x_0 \in S$.

The above definitions can be restated as, f is continuous at x_0 if $\lim_{x \to x_0} f(x) = f(x_0)$.

Now, if we think in terms of sequences, we see that,

f is continuous at x_0,

if for every sequence x_n, that converges to x_0, $f(x_n)$ must converge to $f(x_0)$.

<u>Result</u>: $f : S \to \mathbb{R}$

f is continuous at x_0 iff

$\forall \{x_n\}$ with $x_n \to x_0$,

$f(x_n) \longrightarrow f(x_0)$

Q4 $f, g: \mathbb{R} \to \mathbb{R}$, if f is continuous at x_0 and g is continuous at $f(x_0)$. Prove that $g \circ f$ is continuous at x_0.

Soln: Take any $\{x_n\}$ with $x_n \to x_0$. Then $f(x_n) \to f(x_0)$

$$\therefore\ g(f(x_n)) \to g(f(x_0))$$

By the previous result, $g \circ f$ is continuous at x_0.

$\blacksquare$

Qn Prove that there exists 2022 consecutive integers such that there are exactly 11 primes in them.

Soln: Let $f: \mathbb{N} \to \mathbb{N}$ be defined as $f(n) = \#$ of primes in $\{n, n+1, \ldots, n+2021\}$

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, ...

$\therefore f(1) > 11$

Now, consider $f(2023!+2) = 0$ (Why?)

$f(n)$ and $f(n+1)$ can differ by atmost 1.

$\therefore$ from $n = 1$ to $2023!$

There must have been some $1 \leq n \leq 2023!$ such that $f(n) = 11$. $\blacksquare$

We shall see a similar result for continuous functions from $\mathbb{R}$ to $\mathbb{R}$ in the next class!

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