Introductory Lecture on Combinatorics

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Contents

Part One

1. Lecture 1

1.1 Basic Counting Principles

Theorem 1.1.1 — *The Addition Principle.* If there are n_1 different objects in the first set, n_2 objects in the second set, ..., and n_m objects in the m^{th} set, and if the different sets are disjoint, then the number of ways to select an object from one of the m sets is $n_1 + n_2 + \cdots + n_m$.

Corollary 1.1.2 Let $A_1, A_2, \ldots, A_k$ be any k finite sets, where $k \geq 1$. If the given sets are pairwise disjoint, i.e., $A_i \cap A_j = \emptyset$ for $i, j = 1, 2, \ldots, k, i \neq j$ then

$$\left|\bigcup_{i=1}^{k} A_i\right| = |A_1 \cup A_2 \cup \cdots \cup A_k| = \sum_{i=1}^{k} |A_i|.$$

■ **Example 1.1** How many ways are there to choose 4 distinct positive integer numbers x_1, x_2, x_3, x_4 from the set $S = \{1, 2, \ldots, 499, 500\}$ such that x_1, x_2, x_3, x_4 is an increasing geometric sequence and its common ratio is a positive integer number?
Let $a_1, a_1 q, a_1 q^2, a_1 q^3$ $(a_1, q \in \mathbf{N}_+, q \geqslant 2)$ be the four numbers which are chosen by us, then $a_1 q^3 \leqslant 500, q \leqslant \sqrt[3]{\frac{500}{a_1}} \leqslant \sqrt[3]{500}$. Hence $2 \leqslant q \leqslant 7$, and $1 \leqslant a_1 \leqslant \left[\frac{500}{q^3}\right]$, that is the number of the geometric sequences with the common radio q is $\left[\frac{500}{q^3}\right]$. By the addition principle, the number of the geometric sequences satisfying the conditions is

$$\sum_{q=2}^{7} \left[\frac{500}{q^3}\right] = 62 + 18 + 7 + 4 + 2 + 1 = 94$$

So the answer to the question is 94 . ■

■ **Example 1.2** There are n sticks of length $1, 2, \ldots, n$. How many incongruent triangles can be formed by using three of the given sticks?
Let x, y, z be the lengths of three sticks. Without loss of generality, we may assume that $x < y < z$. These three sticks can form a triangle if and only if x, y, z satisfy the Triangle Inequality; that is,

$x+y>z$. We classify all incongruent triangles by their longest side. For $1 \le k \le n$, define

$$A_k = \{(x,y,z) \mid x,y,z \in \mathbb{Z}, 1 \le x < y < z = k, \text{ and } x+y>z\}$$

Hence, by the Addition Principle, we need to calculate

$$|A_1| + |A_2| + \cdots + |A_n|$$

Under our assumption, $z \ge 3$. Hence $A_1 = A_2 = \emptyset$. If $z = 3$, then $x = 1$ and $y = 2$, and there is no triangle with side lengths $1,2,3$. Thus $A_3 = \emptyset$. Hence $|A_1| = |A_2| = |A_3| = 0$. Now we assume that $k \ge 4$. We consider two cases.

- **Case 1** In this case, we assume that k is even; that is, $k = 2m$ for some integer $m \ge 2$. Because $x < y, x+y > 2x$. Note also that $x+y>z$. We present different arguments for $2x \le z$ and $2x > z$; that is, $1 \le x \le m$ and $m < x$.
  For $1 \le x \le m$, we need $y > z - x = k - x$. Since $k = 2m \ge 2x$ we know that $k - x \ge x$. Hence any y between $k-x+1$ and $z-1 = k-1$, inclusive, will work, implying that there are $(k-1)-(k-x+1)+1 = x-1$ possible values for y
  For $m < x$, the first inequality gives $x+y > 2x > 2m = z$ Thus any y between $x+1$ and $k-1$, inclusive, will work, implying that there are $(k-1)-(x+1)+1 = k-x-1 = 2m-x-1$ possible values for y. Therefore, for $k = 2m$

  $$|A_k| = \sum_{x=1}^{m}(x-1) + \sum_{x=m+1}^{2m-1}(2m-x-1) = \sum_{x=1}^{m}(x-1) + \sum_{i=0}^{m-2} i$$

  implying that

  $$|A_k| = \frac{m(m-1)}{2} + \frac{(m-2)(m-1)}{2} = (m-1)^2.$$

  Note that this formula also works for $m = 1$; that is, $k = 2$.
- **Case 2** In this case, we assume that k is odd; that is, $k = 2m+1$ for some integer $m \ge 2$. For $1 \le x \le m$, we again need $y > z - x = k - x$. This time, $k = 2m+1 > 2x$, so $k - x > x$. As before, y can take on all integer values between $k-x+1$ and $k-1$, inclusive, so there are $(k-1)-(k-x+1)+1 = x-1$ possible values for y. For $m < x$, identical reasoning as in the first case shows that any value of y such that $x < y < z$ will work. Thus there are $(k-1)-(x-1)+1 = k-x-1 = 2m-x$ possible values for y Therefore, for $k = 2m+1$,

  $$|A_k| = \sum_{x=1}^{m}(x-1) + \sum_{x=m+1}^{2m}(2m-x) = \sum_{x=1}^{m}(x-1) + \sum_{i=0}^{m-1} i$$

  implying that

  $$|A_k| = \frac{m(m-1)}{2} + \frac{m(m-1)}{2} = m(m-1).$$

  Note that this formula works also for $m = 0$ and $m = 1$; that is, $k = 1$ and $k = 3$. Now we are ready to solve our problem. If n is odd, then $n = 2p+1$ for some nonnegative integer p. We ave

  $$\begin{aligned}
  &|A_1| + |A_2| + \cdots + |A_n| \\
  =&(|A_1| + |A_3| + \cdots + |A_{2p+1}|) + (|A_2| + |A_4| + \cdots + |A_{2p}|) \\
  =&\sum_{m=0}^{p} m(m-1) + \sum_{m=1}^{p}(m-1)^2 = 2\sum_{m=1}^{p} m^2 - 3\sum_{m=0}^{p} m + p \\
  =&\frac{p(p+1)(2p+1)}{3} - \frac{3p(p+1)}{2} + p = p \cdot \frac{4p^2+6p+2-9p-9+6}{6} \\
  =&\frac{p\left(4p^2-3p-1\right)}{6} = \frac{p(p-1)(4p+1)}{6}
  \end{aligned}$$

If n is even, then $n = 2p$ for some positive integer p. We have

$$
\begin{aligned}
&|A_1| + |A_2| + \cdots + |A_n| \\
=&(|A_1| + |A_3| + \cdots + |A_{2p-1}|) + (|A_2| + |A_4| + \cdots + |A_{2p}|) \\
=&\sum_{m=0}^{p-1} m(m-1) + \sum_{m=1}^{p} (m-1)^2 = \sum_{m=1}^{p-1} m(m-1) + \sum_{m=1}^{p-1} m^2 \\
=&2\sum_{m=1}^{p-1} m^2 - \sum_{m=1}^{p-1} m = \frac{(p-1)(p)(2p-1)}{3} - \frac{p(p-1)}{2} \\
=&p(p-1)\cdot\frac{4p-2-3}{6} = \frac{p(p-1)(4p-5)}{6}
\end{aligned}
$$

Putting the above together, we obtain

$$
\begin{cases}
\frac{p(p-1)(4p+1)}{6(p-1)(4p-5)} \text{ triangles} & \text{if } n \text{ is odd and } n = 2p+1 \\
\frac{p(p-1)(4p-5)}{6} \text{ triangles} & \text{if } n \text{ is even and } n = 2p
\end{cases}
$$

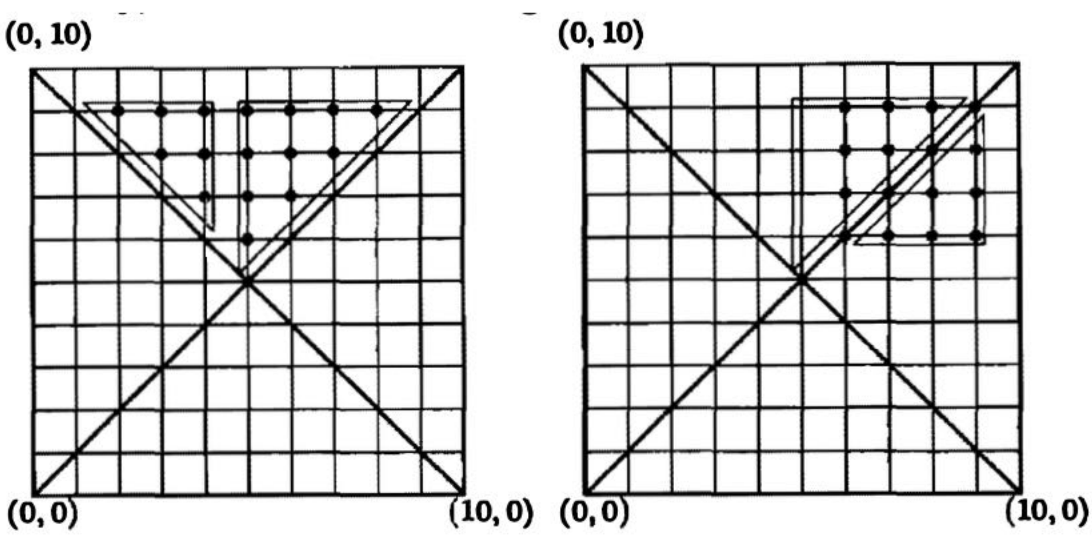

Figure 1.1: For example when k=10,then we can easily get the two necessary equations.

Once the problem was solved for odd $n = 2p+1$, the even case $n = 2p$ could be done just by calculating

$$
\begin{aligned}
&|A_1| + |A_2| + \cdots + |A_{2p}| \\
=&(|A_1| + |A_2| + \cdots + |A_{2p}| + |A_{2p+1}|) - |A_{2p+1}| \\
=&\frac{p(p-1)(4p+1)}{6} - p(p-1) \\
=&\frac{p(p-1)(4p-5)}{6}
\end{aligned}
$$

■

Theorem 1.1.3 — *The Multiplication principle.* Suppose a procedure can be broken into m successive (ordered) stages, with n_1 outcomes in the first stage, n_2 outcomes in the second stage, …, and n_m outcomes in the m^{th} stage. If the composite outcomes are all distinct, then the total procedure has $n_1 n_2 \cdots n_m$ different composite outcomes.

Corollary 1.1.4 Let

$$\prod_{i=1}^{r} A_i = A_1 \times A_2 \times \cdots \times A_r = \{(a_1, a_2, \ldots, a_r) \mid a_i \in A_i, i = 1, 2, \ldots, r\}$$

denote the cartesian product of the finite sets $A_1, A_2, \ldots, A_r$. Then

$$\left| \prod_{i=1}^{r} A_i \right| = |A_1| \times |A_2| \times \cdots \times |A_r| = \prod_{i=1}^{r} |A_i|.$$

■ **Example 1.3** How many 4-digit odd numbers with distinct digits are there?
A 4-digit number is an ordered arrangement of 4 digits (leading zeros not allowed). Since the numbers we want to count are odd, the unit digit can be any one of $1,3,5,7,9$. The tens digit and the hundreds digit can be any one of $0,1,\ldots,9$, while the thousands digit can be any one of $1,2,\ldots,9$. Thus there are 5 choices for the unit digit. Since the digits are distinct, we have 8 choices for the thousands digit, whatever the choice of the unit digit is. Then there are 8 choices for the hundreds digit, whatever the first 2 choices are, and 7 choices for the tens digit, whatever the first 3 choices are. Thus by the multiplication principle, the number of 4 -digit odd numbers with distinct digits is equal to $5 \times 8 \times 8 \times 7 = 2240$. ■

Proposition 1.1.5 Let n be a positive integer, and let $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ be a prime decomposition of n. Then n has

$$(a_1+1)(a_2+1)\cdots(a_k+1)$$

positive integer divisors, including 1 and itself.

Proposition 1.1.6 Let n be a positive integer, and let $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ be a prime decomposition of n. Then there are

$$(2a_1+1)(2a_2+1)\cdots(2a_k+1)$$

distinct pairs of odered positive integers (a,b) such that their least common multiple is equal to n.

■ **Example 1.4** A license plate contains a sequence of three letters of the alphabet followed by a sequence of three digits. How many different license plates can be produced if 0 and O cannot be used at the same time?
Solution: Let S_1 denote the set of license plates with no 0's, and let S_2 denote the set of license plates with no O 's. If $\alpha\beta\gamma - \theta\phi\psi$ is a plate in S_1, then $\theta, \phi, \psi \neq 0$. Consequently, there are no restrictions on α, β, γ; that is, for each of α, β, γ there are 26 choices, while for each of θ, ϕ, ψ there are nine choices. Therefore, $|S_1| = 26^3 \cdot 9^3$. In exactly the same way, $|S_2| = 25^3 \cdot 10^3$ (since the roles of letters and digits are switched). It seems that the answer to the problem is $|S_1| + |S_2| = 26^3 \cdot 9^3 + 25^3 \cdot 10^3$. However, this is not the correct answer. But every step seems logical. Where is the mistake? A more fundamental question is: How do we know whether there is a mistake? We answer the second question first. Let S denote the set of all possible license plates containing a sequence of three letters followed by a sequence of three digits. Then there are 26 choices for each of the three letters and 10 choices for each of the digits. By the multiplication principle, $|S| = 26^3 \cdot 10^3$. It is not difficult to check that

$$|S_1| + |S_2| = 26^3 \cdot 9^3 + 25^3 \cdot 10^3 > 26^3 \cdot 10^3 = |S|$$

This clearly shows that $|S_1|+|S_2|$ is not the desired answer. Now we are going to fix our mistake. We notice that there is some overlap in S_1 and S_2, namely, those license plates that have neither 0 nor O. Let S_3 denote the set of such plates. Then $S_3=S_1\cap S_2$. For each letter of a plate in S_3 there are 25 choices, and for each digit there are nine choices. Thus $|S_3|=25^3\cdot 9^3$. Since each plate in S_3 has been counted twice in S_1 and S_2, the final answer to our problem is

$$|S_1|+|S_2|-|S_3|=26^3\cdot 9^3+25^3\cdot 10^3-25^3\cdot 9^3=17047279$$

■

1.1.1 Permutations

Definition 1.1.1 An ordered arrangement of n distinct objects taking $m(m\leqslant n)$ distinct objects at a time is called a permutation of n distinct objects taking m distinct objects at a time. Since the objects is not repeated, the permutation is also called the permutation without repetition, and the number of "permutation of n distinct objects taking m distinct objects" is denoted by P_m^n or A_m^{n*}, then

$$\mathrm{P}_m^n=n(n-1)(n-2)\cdots(n-m+1)=\frac{n!}{(n-m)!},$$

where $m\leqslant n$, and there is a convention $0!=1$. Especially, when $m=n$, the permutation of n distinct objects taken n distinct objects is called all permutation of n distinct objects. The number of all permutation of n distinct objects is equal to

$$\mathrm{P}_n^n=n(n-1)(n-2)\cdots 2\cdot 1=n!$$

Theorem 1.1.7 Given a set S with $|S|=n$, there are 2^n subsets of S, including the empty set and S itself.

Let A and B be two sets. A map (or mapping or function) f from A to B (written as $f:A\to B$) assigns to each $a\in A$ exactly one $b\in B$ (written $f(a)=b$); b is the image of a. For $A'\subseteq A$, let $f(A')$ (the image of A') denote the set of images of $a\in A'$. If $f(A)=B$, then f is surjective (or onto); that is, every $b\in B$ is the image of some $a\in A$. If every two distinct elements a_1 and a_2 in A have distinct images, then f is injective (or one-to-one). If f is both injective and surjective, then f is bijective (or a bijection or a one-to-one correspondence). A permutation is a change in position within a collection. More precisely, if S is a set, then a permutation of S is a one-to-one function π that maps S onto itself. If $S=\{x_1,x_2,\ldots,x_n\}$ is a finite set, then we may denote a permutation π of S by $(y_1,y_2,\ldots,y_n)$, where $y_k=\pi(x_k)$.

■ **Example 1.5** In a five-team tournament, each team plays one game with every other team. Each team has a 50% chance of winning any game it plays. (There are no ties.) Compute the probability that the tournament will produce neither an undefeated team nor a winless team.

Each team has to play four games. Hence there are 5.4 games if one counts each game twice, once by each of the two teams playing the game. The five teams play a total of $\frac{5\cdot 4}{2}=10$ games. Because each game can have two possible outcomes, by Theorem 1.3, there are 2^{10} possible outcomes for the tournament.

There are five ways to choose an undefeated team. Say team A wins all four of its games. Then each of the remaining six games has two possible outcomes for a total of $2^{10-4}=2^6$ outcomes. Because at most one team can be undefeated, there are $5\cdot 2^6$ tournaments that produce an undefeated team. A similar argument shows that $5\cdot 2^6$ of the 2^{10} possible tournaments produce a winless team.

However, these possibilities are not mutually exclusive. It is possible to have exactly one undefeated team and exactly one winless team in the same tournament. There are ${}_5P_2=5\cdot 4=20$

such two-team permutations. Say team A is undefeated and team B is winless. There are seven (not eight, because A and B play against each other!) games in which either team A or team B or both teams play. The outcomes of these seven games are decided. Each of the remaining three games left has two possible outcomes for a total of $2^{10-7} = 2^3$ tournaments. In other words, $20 \cdot 2^3 = 5 \cdot 2^5$ of the 2^{10} tournaments have both an undefeated team and a winless team. Thus, according to the Inclusion-Exclusion principle, there are

$$2^{10} - 2 \cdot 5 \cdot 2^6 + 5 \cdot 2^5 = 2^5 \left(2^5 - 5 \cdot 2^2 + 5\right) = 2^5 \cdot 17$$

tournament outcomes in which there is neither an undefeated nor a winless team. All outcomes are equally likely; hence the required probability is $\frac{17.2^5}{2^{10}} = \frac{17}{32}$. ■

Theorem 1.1.8 Let n and k be positive integers with $n \geq k$. The total number of permutations of n objects taken k at a time is

$${}_nP_k = n(n-1)\cdots(n-k+1) = \frac{n!}{(n-k)!}$$

where $0! = 1$ and $n! = 1 \cdot 2 \cdots n$, for $n \geq 1$

Corollary 1.1.9 The total number of permutations of n objects is $n!$.

1.1.2 Combinations

Definition 1.1.2 An un-ordered selection of n distinct objects taking $m (m \leqslant n)$ distinct objects at a time is called a combination of n distinct objects taking m distinct objects at a time. Since the objects is not repeated, a combination of n distinct objects taking m distinct objects is also called a combination without repetition. The number of "combination of n distinct objects taking m distinct objects" is denoted

by $\begin{pmatrix} n \\ m \end{pmatrix}$, then

$$\begin{pmatrix} n \\ m \end{pmatrix} = \frac{\mathrm{P}_m^n}{m!} = \frac{n(n-1)(n-2)\cdots(n-m+1)}{m!} = \frac{n!}{m!(n-m)!}$$

■ **Example 1.6** How many 5-digit numbers greater than 21300 are there such that their digits are distinct integers taken from $\{1,2,3,4,5\}$

We divide these 5-digit numbers satisfying the required conditions into 3 types:

The number of 5 -digit number whose ten thousands digit may be any one of 3,4 or 5 is equal to $\mathrm{P}_1^3\mathrm{P}_+^4$.

The number of 5 -digit number whose ten thousands digit be 2 and thousands digit be any one of 3,4 or 5 is equal to $\mathrm{P}_1^3\mathrm{P}_3^3$.

The number of 5 -digit number of ten thousands digit be 2, and thousands digit be 1 is equal to P_3^3.

By the addition principle, the number of 5 -digit numbers satisfying the required conditions is equal to $\mathrm{P}_1^3\mathrm{P}_+^+ + \mathrm{P}_1^3\mathrm{P}_3^3 + \mathrm{P}_3^3 = 96$. ■

1.1.3 Repetitions

Theorem 1.1.10 — *Repeated Permutations.* An ordered arrangement of n distinct objects taking m objects at a time is called a repeated permutation of n distinct objects taken m objects at a time. The number of this repeated permutation is equal to n^m.

Theorem 1.1.11 — *All Permutation of Incomplete Distinct Objects.* Suppose that n objects consist of k distinct objects $a_1, a_2, \dots, a_k$ with repetition numbers $n_1, n_2, \dots, n_m$ $(n_1 + n_2 + \cdots + n_m = n)$ respectively, the all permutation of these n objects is called the all permutations of the incomplete distinct objects. We denote the number of all such permutation by
$\left(\begin{array}{c} n \\ n_1, n_2, \dots, n_k \end{array}\right)$, then $\left(\begin{array}{c} n \\ n_1, n_2, \dots, n_k \end{array}\right) = \frac{n!}{n_1!n_2!\cdots n_k!}$

Let f denote the number of the all permutation satisfying the conditions. If we exchange the same objects in each kind for the mutually distinct objects and rearrange them, then we get $n_1!n_2!\cdots n_k!$ all permutations of n distinct objects. By the multiplication principle, the number of the all permutation of n distinct objects is equal to $f \cdot n_1!\, n_2! \cdots n_k!$. But the number of all permutation of n distinct objects is equal to $n!$. Hence $f \cdot n_1!n_2!\cdots n_k! = n!$. Thus

$$f = \left(\begin{array}{c} n \\ n_1, n_2, \dots, n_k \end{array}\right) = \frac{n!}{n_1!n_2!\cdots n_k!}$$

Theorem 1.1.12 — *Repeated Combination.* An unordered selection of n distinct objects taking m objects (each object may has a finite repletion number) is called a repeated combination. The number of this repeated combination is equal to $\left(\begin{array}{c} n+m-1 \\ m \end{array}\right)$.

Denote the n distinct objects by $1, 2, \cdots, n$. Then repeated combination of n distinct objects taken m objects has the following form: $\{i_1, i_2, \dots, i_m\}$ $(1 \leqslant i_1 \leqslant i_2 \leqslant \cdots \leqslant i_m \leqslant n)$. Since the selections could be repeated, so that the equality holds. Set $j_1 = i_1, j_2 = i_2 + 1, \cdots, j_m = i_m + (m-1)$, then $1 \leqslant j_1 < j_2 < \cdots < j_m \leqslant n + m - 1$, and the $\{j_1, j_2, \dots, j_m\}$ is just the combination without repetition of $n + m - 1$ distinct objects: $1, 2, \dots, n + m - 1$ taken m distinct objects.

Hence the number of the required repeated combination equals $\left(\begin{array}{c} n+m-1 \\ m \end{array}\right)$.

Theorem 1.1.13 — *Multiple Combination.* Let's classify n distinct objects into $k (k \leqslant n)$ distinct kinds, such that there are n_i objects in i^{th} kind $(i = 1,, 2, \dots, k, n_1 + n_2 + \cdots + n_k = n)$. Then the number of the classify ways is equal to $\left(\begin{array}{c} n \\ n_1, n_2, \dots, n_k \end{array}\right) = \frac{n!}{n_1!n_2!\cdots n_k!}$.

Since the number of ways of the n distinct objects taken n_1 distinct objects is equal to $\left(\begin{array}{c} n \\ n_1 \end{array}\right)$. Then, the number of ways taking n_2 distinct objects from the residual $n - n_1$ distinct objects is equal to $\left(\begin{array}{c} n-n_1 \\ n_2 \end{array}\right)$. If we continue like this and invoke the multiplication principle, we find that the number of distinct partitioned kinds equals

$$\begin{aligned}&\binom{n}{n_1}\binom{n-n_1}{n_2}\cdots\binom{n-n_1-n_2-\cdots-n_{k-1}}{n_k}\\=&\frac{n!}{n_1\,(n-n_1)!}\cdot\frac{(n-n_1)!}{n_2!\,(n-n_1-n_2)!}\cdots\frac{(n-n_1-\cdots-n_{k-1})!}{n_k!\,(n-n_1-\cdots-n_k)!}\\=&\frac{n!}{n_1!n_2!\cdots n_k!}\end{aligned}$$

■ **Example 1.7** In how many ways can one chose 10 paper currencies from the bank and the volumes of these paper currencies are 1 Jiao, 5 Jiao, 1 Yuan, 5 Yuan, 10 Yuan 50 Yuan and 100 Yuan respectively? (Remark: The Jiao and Yuan are the units of money in China.)
We are asked to count the repeated combinational number of ways to take 10 paper currencies from 7 distinct paper currencies. Using the formula of repeated combinatorial number, we get that the number of required distinct ways equals

$$\binom{7+10-1}{10}=\binom{16}{6}=\frac{16\times15\times14\times13\times12\times11}{1\times2\times3\times4\times5\times6}=8008$$

■

■ **Example 1.8** Suppose that 3 red-flags, 4 blue-flags and 2 yellow-flags are placed on 9 numbered flagpoles in order (every flagpole hangs just one flag). How many distinct symbols consist of these flags are there?
Using the formula of all permutation number of incomplete distinct objects, we get that the number of distinct symbols

$$\text{equals}\ \binom{9}{3,4,2}=\frac{9!}{3!4!2!}=1260$$

■

■ **Example 1.9** How many are there to choose 3 pairs of players for the doubles from $n(\geqslant 6)$ players.
The number of taking 6 players from n distinct players equals $\binom{n}{6}$. The 6 players is classified into three groups such that each group contains exactly 2 players and the number of methods equals $\binom{6}{2,2,2}$, but the 3 groups are unordered, so the number required ways is equal to

$$\frac{\binom{n}{6}\binom{6}{2,2,2}}{3!}=\frac{n!}{6!(n-6)!}\cdot\frac{6!}{2!2!2!}\cdot\frac{1}{3!}=\frac{n!}{48(n-6)!}$$

■

1.1.4 Some Other Basic Variants

Theorem 1.1.14 — *Circular Permutation of Distinct Elements.* If we arrange the n distinct objects in a circle, then this permutation is called a circular permutation of n distinct objects. The number of circular permutation of n distinct objects equals $\frac{\mathrm{P}_n^n}{n}=(n-1)!$.

Since n linear permutations $A_1A_2\cdots A_{n-1}A_n, A_2A_3\cdots A_nA_1,\ldots,A_nA_1\cdots A_{n-2}A_{n-1}$ give rise to the same circular permutation and there are P_n^n linear permutations. Thus the number of circular permutations of n distinct objects equals $\frac{\mathrm{P}_n^n}{n}=(n-1)!$

Theorem 1.1.15 — *Number of Necklace.* Suppose that a necklace consists of n distinct beads which are arranged in circle, then the number of distinct necklaces is 1 (if $n=1$ or 2) or $\frac{1}{2}\cdot(n-1)!($ if $n \geqslant 3)$.

If $n=1$ or 2, then the number of necklace is 1. Assume that $n \geqslant 3$. Since a necklace can be rotated or turned over without any change, the number of necklaces is one-half of the number of circular permutation of n distinct objects, i. e. $\frac{1}{2}\cdot(n-1)!$

■ **Example 1.10** How many ways are there to arrange 6 girls and 15 boys to dance in a circle such that there are at least two boys between any two adjacent girls?

First, for every girl, we regard two boys as her dancing partner such that one is at the left of this girl and another is at the right. Since 6 girls are distinct, we can select 12 boys from 15 boys in P_{12}^{5} ways. Next, every girl and her two dancing partners are considered as a group, each of residual $15-12=3$ boys are also considered as a group. Thus the total of groups is 9 , and we can arrange them in a circle in $(9-1)!=8!$ ways. By the multiplication principle, the number of permutations satisfying the conditions equals $\mathrm{P}_{12}^{5}\cdot 8!=\frac{15!\cdot 8!}{3!}$ ■

Theorem 1.1.16 — *The number of Solutions of The Indefinite Equation.* The number of non-negative integer solutions $(x_1,x_2,\ldots,x_m)$ of the indefinite equation $x_1+x_2+\cdots+x_m=n\,(m,n\in\mathbf{N}_+)$ is equal to $\binom{n+m-1}{m-1}=\binom{n+m-1}{n}$

We consider that each nonnegative integer solution $(x_1,\ x_2,\ldots,x_m)$ of the equation $x_1+x_2+\cdots+x_m=n\,(m,n\in\mathbf{N}_+)$ corresponds to a permutation of n circles "O" and $m-1$ bars " | " : $\underbrace{\mathrm{OO}\cdots\mathrm{O}}_{x_1}|\underbrace{\mathrm{OO}\cdots\mathrm{O}}_{x_2}|\cdots|\underbrace{\mathrm{OO}\cdots\mathrm{O}}_{x_m}$ Where x_1 is the number of circles "O" at the left of first bar " | ",x_{i+1} is the number of circles "O" between the i^{th} bars " | " and the $(i+1)^{\text{th}}$ bars " | ",$\cdots$,x_m is the number of circles "O" at the right of the $(m-\ 1\ )$ th bar "|". Since the correspondence is an one-to-one correspondence, the number of nonnegative integer solutions $(x_1,,\ x_2,\ldots,x_m)$ of the indefinite equation $x_1+x_2+\cdots+x_m=n(m,,\ n\in\mathbf{N}_+)$ equals the number of the permutations of n circles " O " and $(m-1)$ bars " | ", i.e. $\binom{n-m+1}{m-1}=\binom{n+m-1}{n}$. Remark The number of nonnegative integer solutions $(x_1,,\ x_2,\ldots,x_m)$ of the indefinite equation $x_1+x_2+\cdots+x_m=n(m,,\ n\in\mathbf{N}_+)$ is equal to the number of the repeated combinations from n distinct objects taken m objects (each object may has a finite repletion number).

Theorem 1.1.17 The number of positive integer solutions $(x_1,x_2,\ldots,,\ x_m)$ of the indefinite equation $x_1+x_2+\cdots+x_m=n\,(m,n\in\mathbf{N}_+,n\geqslant m\,)$ equals $\binom{n-1}{m-1}$.

Setting $y_i=x_i-1(i=1,2,\cdots,m)$, we get $y_1+y_2+\cdots+\ y_m=n-m$. Thus the number of positive integer solutions $(x_1,,\ x_2,\ldots,x_m)$ of the indefinite equation $x_1+x_2+\cdots+x_m=n(m,,\ n\in\mathbf{N}_+,n\geqslant m)$ equals number of nonnegative integer solutions $(y_1,,\ y_2,\ldots,y_m)$ of the indefinite equation $y_1+y_2+\cdots+y_m=n-m$, i. e. $\binom{(n-m)+m-1}{m-1}=\binom{n-1}{m-1}$.

Exercise 1.1 Let p be a prime integer. Prove that $\binom{2p}{p} - 2$ is divisible by p^2 ■

Exercise 1.2 Let n, m be non-negative integers such that $n \geq m$. Prove that

$$\binom{n+m}{m} = \binom{n}{0}\binom{m}{m} + \binom{n}{1}\binom{m}{m-1} + \binom{n}{2}\binom{m}{m-2} + \cdots + \binom{n}{m}\binom{m}{0}$$

■

Exercise 1.3 Show that the number of lists $(b_1, b_2, \ldots, b_{m+1})$ of positive integers such that $b_1 + b_2 + \cdots + b_{m+1} = n+1$ is $\binom{n}{m}$. ■

Exercise 1.4 A spider has 8 feet, 8 different shoes and 8 different socks. Find the number of ways in which the spider can put on the 8 socks and the 8 shoes (considering the order in which it puts them on). The only rule is that to put a shoe on the spider must already have a sock on that foot. ■

Exercise 1.5 Ivan and Alexander write lists of integers. Ivan writes all the lists of length n with elements $a_1, a_2, \ldots, a_n$ such that $|a_1| + |a_2| + \cdots + |a_n| \leq k$. Alexander writes all the lists with length k with elements $b_1, b_2, \ldots, b_k$ such that $|b_1| + |b_2| + \cdots + |b_k| \leq n$. Prove that Alexander and Ivan wrote the same number of lists. ■

Exercise 1.6 Show that if $n \geq k \geq r \geq s$, then

$$\binom{n}{k}\binom{k}{r}\binom{r}{s} = \binom{n}{s}\binom{n-s}{r-s}\binom{n-r}{k-r}.$$

■

Exercise 1.7 A number of persons seat at a round table. It is known that there are 7 women who have a woman to their right and 12 women who have a man to their right. We know that 3 out of each 4 men have a woman to their right. How many people are seated at the table? ■

Exercise 1.8 Find the number of ways to place 3 rooks on a 5×5 chess board so that no two of them attack each other.

■

Exercise 1.9 A square board with side-length of 8 cm is divided into 64 squares with side-length of 1 cm each. Each square can be painted black or white. Find the total number of ways to color the board so that every square with side-length of 2 cm formed with 4 small squares with a common vertex has two black squares and two white squares. ■