Handwritten Notes on
Number Theory & Combinatorics

www.fractionshub.com

Diganta Bhattacharya

NUMBER THEORY
&
COMBINATORICS

INTRO TO LECTURE 1 (PART 2)

Lecture notes :

Diganta B. B.Stat ISI Kolkata.
M.Stat. Currently pursuing.

4 LECTURES + 4 EXAMS

→ Topic based. (Not model exam)

+ 10 model exams on full syllabus.

NUMBER THEORY

1. Well Ordering Principle
2. Mathematical Induction
3. Divisibility & Identities
4. GCD & LCM + Primes
5. Divisibility Tests & Congruences
6. Number Theoritic functions

ADVANCED TOPICS:

1. RESIDUES
2. FERMAT AND EULER'S WORK
3. CHINESE REMANDER & APPLICATIONS

SUGGESTED READS:

Problem Solving Strategies
— ARTHUR ENGEL

COMBINATORICS

1. PERMUTATIONS & COMBINATIONS
2. Binomial & Multinomial coefficients
3. Pigeonhole Principle
4. Principle of Inclusion-Exclusion
5. Recurrence Relations

ADVANCED TOPICS:

1. GENERATING FUNCTIONS
2. RAMSEY NUMBERS
3. PARTITIONS

SUGGESTED READS:

Principles and Techniques in Combinatoris
— CHEN CHUAN CHONG & KOH KHEE MENG

So, in each class we will have a topic revision, and questions and answers on those. To give you an idea of the type of problems we will be doing →

Let us start with some basic counting →

In how many ways can a committee of 5 be formed from a group of 11 people consisting of 4 teachers and 7 students if →

(i) There is no restriction in selection?
(ii) A committee must include exactly 2 teachers?
(iii) A committee must include at least 3 teachers?
(iv) A particular teacher and a particular student cannot be both in the committee?

(i) It is simply $\binom{11}{5} = \frac{11!}{5!6!} = 462$.

(ii) Select 2 teachers from 4, then select $(5-2) = 3$ students from 7. Teachers that got allocated

No of ways → $\binom{4}{2}\binom{7}{3} = 6 \times 35 = 210$

(iii) 2 cases ⟶ 3 teachers in the committee → $\binom{4}{3}\binom{7}{2} = 84$
4 teachers in the committee $\binom{4}{4}\binom{7}{1} = 7$

So total no. of ways → $84 + 7 = 91$

(iv) Take a particular teacher say T & a particular student S. So what is the number of ways we can form a committee of 5 which includes both T and S.

⇒ We take union of $\{T, S\}$ and a subset of 3 from the remaining 9.

⇒ No of ways to form a committee of 5 including T and S is $\binom{9}{3} = 84$.

⇒ No of ways to form committee of 5 which does not include both T & S $= \binom{11}{5} - \binom{9}{3} = 378$

Pretty easy right? Let's pick up the pace a bit. Do not worry, we will get to difficult problems.

EXCERCISE: A new sequence $\{a_n\}$ is obtained from the sequence of the positive integers $\{1, 2, \ldots\}$ by deleting all multiples of 3 or 4 except multiples of 5. What is the 2009th term of this sequence?

HINT & ANSWER: Say we denote $A_K = \{k \mid k \in \{1, \ldots, n\}, k$ is divisible by $i\}$ where $i = 3, 4, 5$ here then the set of Numbers not deleted →

$$(A_3^c \cap A_4^c \cap A_5^c) \cup A_5 \longrightarrow \text{Answer} = \underline{3347}$$

Digression:

Priciple of Inclusion Exclusion is familiar right?

What about: (A different form)

If S is a finite set, $A_i \subset S$ $(i = 1, 2, \ldots, n)$ and the complement of A_i in S is say A_i^c then →

$$|A_1^c \cap A_2^c \cap \ldots A_n^c| = |S| - |A_1 \cup A_2 \ldots \cup A_n|$$

$$= |S| - \sum_{i=1}^{n} |A_i| + \sum_{1 \le i < j \le n} |A_i \cap A_j| - \ldots + (-1)^n |A_1 \cap A_2 \ldots \cap A_n|.$$

fancy name

SIEVE FORMULA /
SUCCESSIVE SWEEP PRINCIPLE.

Applications

EXCERCISE: Determine the no. of positive integers less than 1000 which is not not divisible by 5 or 7?

$$|A_5^c \cap A_7^c| = |S| - |A_5| - |A_7| + |A_5 \cap A_7| = 999 - 199 - 142 + 28 = 686.$$

Still pretty easy right? Okay let's move on →

In a school there are b teachers and c students satisfying the following conditions: →

(1) Each teacher teaches exactly k students.

(2) for any two distinct students, there are exactly h teachers teaching them.

Prove that $b/h = \frac{c(c-1)}{k(k-1)}$.

If a teacher T_r teaches two students S_i, S_j $(i \neq j)$ then → (T_r, S_i, S_j) → a triple denoting this combination. Also let us take the number of all such triples (T_r, S_i, S_j) as M.

If the teacher T_r teaches k students as given in the problem, then there are $\binom{k}{2}$ triples (T_r, S_i, S_j) containing T_r and there are exactly b many choices for T_r.

$$\Rightarrow M = b.\binom{k}{2} \quad \dots (i)$$

Now, for any two students S_i, S_j $(i \neq j)$ there are exactly h teachers who teach them. So there are h triples (T_r, S_i, S_j) containing S_i, S_j $(i \neq j)$ & there are exactly $\binom{c}{2}$ ways to select the students S_i, S_j $(i \neq j)$ and → $M = h.\binom{c}{2} \quad \dots (ii)$

From (i) & (ii) $b\binom{k}{2} = h\binom{c}{2} \Rightarrow \frac{b}{h} = \frac{\binom{c}{2}}{\binom{k}{2}} = \frac{c(c-1)}{k(k-1)}$.

fancy name → COUNTING IN TWO WAYS

Okay. Now we are getting there. Two more combinatorics problems to go. A small exercise for you to check your knowledge.

EXERCISE : Suppose that a group of n persons satisfy the following conditions:

For any $n-2$ persons, the number of pairs of mutually acquainted persons are equal and equals 3^k for some positive integer k. Find all possible values of n.

- Hint : Define mutually acquainted persons as two people who know each other. What should be the total number of pairs of mutually acquainted persons?

Now let's do something both tricky and easy at the same time.

Prove that $$\sum_{k=0}^{n} \binom{n}{k} 2^k \binom{n-k}{\left[\frac{n-k}{2}\right]} = \binom{2n+1}{n}.$$

Now consider the following problem: —

Say there are $2n$ students → n boys and n girls in a class with their teacher T. Let $g_1, \dots g_n$ denote all the girls and $b_1, \dots, b_n$ denote all the boys, for $1 \le i \le n$ (g_i, b_i) are paired. The class has n tickets to a soccer game. What is the number of ways n people from this class can go to a game? ⟶ Easy right? $\binom{2n+1}{n}$ → $2n+1$ people n tickets.

Now look back at the previous problem :→

$$\sum_{k=0}^{n} \binom{n}{k} \cdot 2^k \cdot \binom{n-k}{\left[\frac{n-k}{2}\right]} = \binom{2n+1}{n}$$

↳ Answer to the question

How many ways to go to the SOCCER GAME?

let us answer this in a different manner

Fix an integer k such that $1 \le k \le n$, and we find k pairs from n pair of students, and give each pair 1 ticket. There are a total of $\binom{n}{k} \cdot 2^k$ ways to find k pairs and pick one student from each pair to go to the game. We have → $n-k$ tickets left and $n-k$ pairs of students left. We pick $\left[\frac{n-k}{2}\right]$ many pairs from the remaining and give each pair 2 tickets.

No. of ways one can do that → $\binom{(n-k)}{\left[\frac{(n-k)}{2}\right]}$. So,

we have assigned $S = k + 2 \cdot \left[\frac{n-k}{2}\right]$ tickets.

If $n-k$ is odd, then $S = n-1$ and we assign the last ticket to teacher T. If $n-k$ is even, then $S = n$ & we have assigned all the tickets then. Now, as k ranges from 1 to n, we obtain all possible ways of assigning n-tickets. So there are

$\sum_{k=1}^{n} \binom{n}{k} \cdot 2^k \cdot \binom{n-k}{\left[\left(\frac{n-k}{2}\right)\right]}$ ways to find n people to go to the game.

Thus → $\sum_{k=1}^{n} \binom{n}{k} \cdot 2^k \cdot \binom{n-k}{\left[\frac{n-k}{2}\right]} = \binom{2n+1}{n}$.

EXCERCISE: There are n distinct points on a plane. Prove that the no. of point pairs with unit distance is smaller than

→ $n/4 + \frac{\sqrt{2}}{2} \cdot n^{1/2}$.

→ MORE TO BE DISCUSSED IN CLASSES FOR COMBINATORICS.

Now, let us begin with some elementary questions in Number theory: Let me start with my ISI-Interview Question.

Suppose there are 1000 doors, and there are 1000 students. The doors are all closed initially. The first student comes and opens all doors. The second student comes and closes all even numbered doors. The third student comes and closes an open door or opens a closed door, whose numbers are 3, 6, 9, So, the 1000th person comes and changes the state of the 1000th door only. How many doors are open after all students are done? Hint: Think about those nos. which have odd no. of factors.

Interesting Observation: Perfect squares have odd no. of factors.

↳ Why? [It has a square root integer!]

↳ So the previous answer should be the no. of square integers < 1000.

$25 = 1 \times 25$
$\quad\;\; \underline{5} \times \underline{5}$
$\quad\;\; x \times x$

$26 = 1 \times 26$
$\quad\;\; 2 \times 13$
$\quad\;\; x \times y$.

Pretty easy right? Let us take some examples involving primes.

Let $p > 5$ be a prime number and let →

$$X = \{p - n^2 \mid n \in \mathbb{N},\ n^2 < p\}.$$

prove that X contains two distinct elements x, y such that $x \neq 1$ and x divides y.

Let us take m such that $m^2 < p < (m+1)^2$ then we can say →

$p = k + m^2$ for some $1 \le k \le 2m$.

$p - (m-k)^2 = k(2m+1-k)$ & $p - m^2 = k$. So →

Now, $p-m^2 \mid p-(m-k)^2$ & $k \neq m$ since p is a prime number, otherwise

↳ $p - m^2 \mid p$ → not possible.

And also, $m - k < m$

& $-m + k \neq m$ since, p is not composite.

We just need to check when $k = 1$. Then since m is even,

$p - (m-1)^2 \mid p - 1$ & hence we are done. ∎

Now try to answer two very intriguing questions about primes.

① Can you always find a range of consequtive numbers of any specified length such that no number in the range is a prime?

② Can you always find a range of consequtive numbers with exactly 23 primes?

Surprisingly or not, both the answers are yes. Why?.

After working with primes, we face problems on divisibility.

Prove that if k is odd then $2^{n+2} \mid k^{2^n} - 1$ for all natural number n.

For $n=1$ it is true, as $k^2-1=(k-1)(k+1)$ is divisible by 8 for any odd no. k since, both $(k-1)$, $(k+1)$ will be divisible by 2 & one of them would be by 4, & the rest follows from induction.

Digression: $n!$ (divides product of n consequtive nos.).
$\hookrightarrow n! \mid (x+1)(x+2)\cdots(x+n) \quad \forall\, x \in \mathbb{N}$.

An integer n is called good if we can write the following:

$$n = a_1 + a_2 + \cdots + a_k$$

where a_i's are positive integers not necessarily distinct, satisfying $\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_k} = 1$. Given that all the integers 33 through 73 are good. Prove that every integer ≥ 33 is good. $\longrightarrow$ LECTURE 1 starts with this.