# Handwritten Notes on Polynomials

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## Demo Lecture - by Soutim Das

## TOPIC: Polynomials

"Polynomials" is one of the most crucial topics for all math competitions. ISI and CMI entrances are no exception. In this demo lecture, we shall try to provide a nearly exhaustive categorisation of the sub-topics from which problems on polynomials are asked in ISI/CMI exams. We shall mainly provide insightful examples corresponding to each such subtopic and shall take a close look at the problem solving techniques used in solving them.

## ① System of Equations:

Quite unexpected for a start, but here's an illustrative example regarding how QUARTIC polynomials can be cleverly used to simplify quite difficult problems:

<u>Problem</u>: Solve the following system of equations for $a, b, c, d \in \mathbb{R}$.

$$a+b = 9$$
$$ab+c+d = 29$$
$$ad+bc = 39$$
$$cd = 18$$

<u>Caution</u> – If you try to solve the problem just by algebraic manipulation tools like elimination and such, you might find the problem excruciatingly lengthy and messy.

Solution: Quartic polynomials, as many of you might be knowing, are polynomials of degree 4. So, how to use them in the context of this problem?

Notice that, if we can frame a monic (leading term coefficient = 1) polynomial of degree 4 whose non-unit (i.e., $\neq 1$) coefficients are the expressions on the LHS of our problem statement, we might make some progress.

So, consider the following polynomials:

$$P_1(x) := x^2 + ax + c$$

$$P_2(x) := x^2 + bx + d$$

Let us define our quartic polynomial as follows:

$$P(x) := P_1(x) \cdot P_2(x)$$

$$= (x^2+ax+c)(x^2+bx+d)$$

Trivial algebraic computation yields:

$$P(x) = x^4 + (a+b)x^3 + (ab+c+d)x^2 + (ad+bc)x + cd$$

And voila! We have indeed achieved a great deal!

Now consider another quartic polynomial:

$$Q(x) := x^4 + 9x^3 + 29x^2 + 39x + 18$$

We are almost done. According to the problem, the following equality holds (why?):

$$P(x) = Q(x)$$

$$\Rightarrow \quad \begin{aligned} &x^4 \\ &+ (a+b)x^3 \\ &+ (ab+c+d)x^2 \end{aligned} = \begin{aligned} &x^4 \\ &+ 9x^3 \\ &+ 29x^2 \end{aligned}$$

$$
\begin{aligned}
&+ (ad+bc)x \qquad && + 39x \\
&+ cd && + 18
\end{aligned}
$$

We know that $P(x)$ can be factored into 2 monic quadratic polynomials. So, we try to similarly factorize $Q(x)$ into two monic quadratics. Observe that:

$$Q(x) = (x+1)(x+2)(x+3)(x+3)$$

So, $Q(x)$ can be factored into a pair monic quadratic in 4 (Why?) different ways (considering ordered pairs). Now, can you finish the problem from here?

## ② Existence of Roots:

This sub-topic is even more popular in exams due to the wide range of problems that can be asked from it. The most handy (as well as elegant) problem solving technique used here is the bonafide Intermediate Value Theorem (IVT) from Calculus.

**Statement of IVT:** Consider an interval $I = [a,b]$ of real numbers and a continuous function $f: I \longrightarrow \mathbb{R}$. Suppose $f(a) \neq f(b)$. Without loss of generality let $f(a) < f(b)$. Then for any $c \in (f(a), f(b))$, $\exists$ (there exists) at least one $x_0 \in (a,b)$ such that $f(x_0) = c$.

Now, we are equipped to face the following problem:

Problem: Two players, Player 1 and Player 2, play the following game:

They take turns choosing a coefficient $a_i$ from the following polynomial:

$$P(x) := x^{10} + a_9 x^9 + \dots + a_1 x + 1$$

Then they replace $a_i$ with an arbitrary real number. The game ends after nine moves when all the $a_i$'s have been replaced. The first player to choose, i.e., Player 1, wins the game if the resulting polynomial has no real root, while the second player wins if the resulting polynomial has at least one real root.

Do any of the 2 players have a winning strategy. If yes, then: (i) who has the strategy? and (ii) what is the strategy?

<u>Solution:</u> We shall show that player 2 has a winning strategy.

Notice that, amongst $a_1, a_2, \ldots, a_9$, 5 coefficients correspond to odd powers of $x$ and the remaining 4 correspond to even powers of $x$.

If Player 1 chooses $a_{k_1}$ in his first move, $k_1$ odd, then Player 2 <u>should</u> choose $a_{\ell_1}$ in his first move, $\ell_1$ even, and vice-versa. Player 2 replaces $a_{\ell_1}$ with any value he wants.

Player 2 <u>should</u> repeat the procedure for $a_{\ell_2}$ and $a_{\ell_3}$, i.e., his 2nd and 3rd moves.

Now, after 7 moves, when it is Player 2's final turn, let the unchosen coefficients be $a_m$ and $a_n$. Notice that at least one of $a_m$ and $a_n$ is odd. (Why?)

Let $R(x)$ denote the part of $P(x)$ that has already been determined by the first seven choices. Then:

$$P(x) = R(x) + a_m x^m + a_n x^n$$

Now, we have reached the crux of the problem.

Case 1: $m$ is odd, $n$ is even.

Observe that:

$$P(1) = Q(1) + a_m + a_n$$

$$P(-1) = Q(-1) - a_m + a_n$$

$$\Rightarrow P(1) + P(-1) = Q(1) + Q(-1) + 2a_n$$

Player 2 <u>should</u> choose $a_n =$
$-\frac{1}{2}(Q(1) + Q(-1))$ in this last move.

Then: $P(1) + P(-1) = 0$ !

Can you use IVT now to guarantee that Player 2 wins? Try it out!

## Case 2: $m, n$ are both odd.

Observe that:

$$P(2) = Q(2) + 2^m a_m + 2^n a_n$$

$$P(-1) = Q(-1) - a_m - a_n$$

$$\text{or, } 2^n P(-1) = 2^n Q(-1) - 2^n a_m - 2^n a_n$$

$$\Rightarrow 2^n P(-1) + P(2) = 2^n Q(-1) + Q(2) + (2^m - 2^n) a_m$$

Player 2 <u>should</u> choose $a_m = -\left(2^n Q(-1) + Q(2)\right) / (2^m - 2^n)$ in his last move.

Then: $2^n P(-1) + P(2) = 0$ !

Again, use IVT to prove that Player 2 wins!

So, no matter what Player 1 does, Player 2 always has a winning strategy!

<u>Remark</u>: In this type of strategy-finding problems, looking at the first move/last move or both is often of great help!

That's all for now...
Remaining sub-topics and related problem solving techniques to follow in upcoming lectures! Stay tuned!