Notations: In the following, $\mathbb{N} = \{1, 2, 3, \cdots\}$ denotes the set of natural numbers, $\mathbb{R}$ denotes the set of real numbers.

1 Sample questions

1. Let $S = \{1, 2\} \subseteq \mathbb{R}$. Consider the function $f : \mathbb{R} \to \mathbb{R}$, given by $f(x) = \inf\{|x - y| : y \in S\}$. Then

   (A) f is not continuous.

   (B) f is continuous but not differentiable only at 1.

   (C) f is continuous but not differentiable only at 1 and 2.

   (D) f is continuous but not differentiable only at 1, 3/2 and 2.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be twice differentiable with $f''(x) > 0, \forall x \in \mathbb{R}$. Then which of the following is true?

   (A) $f(x) = 0$ has exactly two solutions in $\mathbb{R}$.

   (B) $f(x) = 0$ has a positive solution if $f(0) = 0$ and $f'(0) = 0$.

   (C) $f(x) = 0$ has no positive solution if $f(0) = 0$ and $f'(0) > 0$.

   (D) $f(x) = 0$ has no positive solution if $f(0) = 0$ and $f'(0) < 0$.

3. Consider the following statements.

   (a) If f is uniformly continuous on disjoint closed intervals $I_1, I_2, \ldots, I_n$, then f is uniformly continuous on $\cup_{j=1}^{n} I_j$.

   (b) If f is uniformly continuous on disjoint open intervals $I_1, I_2, \ldots, I_n$, then f is uniformly continuous on $\cup_{j=1}^{n} I_j$.

   Which of the following is true.

   (A) Both (a) and (b) are correct.

   (B) Both (a) and (b) are incorrect.

   (C) Only (a) is correct.

   (D) Only (b) is correct.

4. The function $f(x) = a_0 + a_1|x| + a_2|x|^2 + a_3|x|^3$ is differentiable at $x = 0$

   (A) for no values of a_0, a_1, a_2, a_3.

   (B) for any value of a_0, a_1, a_2, a_3.

   (C) only if $a_1 = 0$.

   (D) only if both $a_1 = 0$ and $a_3 = 0$.

5. Let $f(x) = 2 + \frac{1}{3}(x-2)$. Define a sequence of functions $\{f_n\}_n$ from $\mathbb{R}$ to $\mathbb{R}$ as:

$$f_1(x) = f(x),\ \ f_n(x) = f_{n-1}(f(x)), n \geq 2.$$

Which of the following statements is not true?

(a) $\lim_{n\to\infty} f_n(x)$ does not exist for any x.

(b) $\lim_{n\to\infty} f_n(x)$ exists only for $|x-2| < 1$.

(c) $\lim_{n\to\infty} f_n(x) = 2$ for all x.

(d) $\lim_{n\to\infty} f_n(x) = \frac{2}{3}$ for all x.

6. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a twice differentiable function such that f'' is continuous and $f(0) = 1,\ f'(0) = 3, f''(0) = 5$. Which of the following limit exists?

(a) $\lim_{h\to 0} \frac{f(2h)-f(3h)+2h}{h^2}$.

(b) $\lim_{h\to 0} \frac{f(2h)-f(-5h)-21h}{h^2}$.

(c) $\lim_{h\to 0} \frac{f(3h)+f(-3h)-2}{h}$.

(d) $\lim_{h\to 0} \frac{f(h)+f(2h)-2f(3h)+2h}{h^2}$.

7. Let $f : [0,1] \to [0,1]$ be a continuous function and let $f_n(x) = f(x)^n$. Suppose the sequence $\{f_n\}_n$ converges uniformly on $[0,1]$. Which of the following statements is necessarily false?

(a) $\sup_{x\in[0,1]} f(x) = 1$.

(b) $\lim_{n\to\infty} f_n(x) = 0$ for all $x \in [0,1]$.

(c) $\lim_{n\to\infty} (\int_0^1 f_n(x)dx)(1 - \int_0^1 f_n(x)dx) = 0$.

(d) $\inf_{x\in[0,1]} f(x) < 1,\ \sup_{x\in[0,1]} f(x) = 1$.

8. Define a sequence $\{x_n\}_n$ by

$$x_0 = 1, x_n = 2x_{n-1} + 1 \text{ if } n \text{ is odd},\ 3x_{n-1} + 2 \text{ if } n \text{ is even}.$$

Then

(a) $\lim_{n\to\infty} \frac{\log(x_n)}{n}$ does not exist.

(b) $\lim_{n\to\infty} \frac{\log(x_n)}{n} = \frac{1}{2}\log 6$.

(c) $\lim_{n\to\infty} \frac{\log(x_n)}{n} = \log 5$.

(d) $\lim_{n\to\infty} \frac{\log(x_n)}{n} = \frac{5}{2}$.

9. Let $f : [0,\infty) \to [0,\infty)$ be a continuous function such that $\int_0^\infty f(x)dx < \infty$. Which of the following is necessarily true?

(a) $\int_0^\infty f(x)^2 < \infty$.

(b) $\lim_{x\to\infty} f(x) = 0$.

(c) $\liminf_{x\to\infty} f(x) = 0$.

(d) $\sum_n f(n) < \infty$.

10. V is a finite dimensional vector space and P is a non-zero linear map from V to V such that $P^2 = P$. Which of the following statements is not necessarily true?

(a) $\mathrm{Ran(P)} \cap \mathrm{Ran(I-P)} = \{0\}$.

(b) $(I-P)^2 = I - P$.

(c) P is diagonalisable.

(d) P is invertible.

11. Suppose T is a linear map from a three dimensional real vector space V to itself. The characteristic polynomial of T is $(x-1)(x-2)(x-3)$. If W is a two-dimensional subspace of V such that $T(W) \subseteq W$, which of the following polynomials cannot be the characteristic polynomial for the linear map $T : W \to W$?

(a) $(x-2)$.

(b) $(x-1)(x-2)$.

(c) $(x-2)(x-3)$.

(d) $(x-1)(x-3)$.

12. Let a, b, c be positive reals such that $b^2 + c^2 < a < 1$. Consider the 3×3 matrix $M = \begin{pmatrix} 1 & b & c \\ b & a & 0 \\ c & 0 & 1 \end{pmatrix}$

(A) All the eigenvalues of M are negative real numbers.

(B) All the eigenvalues of M are positive real numbers.

(C) M can have positive as well as negative eigenvalues.

(D) M can have complex eigenvalues with nonzero imaginary part.

13. Suppose G is a group in which $x^2 = 1$ for every $x \in G$. Which of the following is necessarily true?

(a) G is finite.

(b) G is infinite.

(c) G is commutative.

(d) G has no non-trivial normal subgroup.

14. Suppose G is a group and $a, b \in G$. Suppose H is a subgroup of G such that $Ha = Hb$. Which of the following is necessarily true?

   (a) There exists a positive integer n such that $a^n, b^n \in H$.
   (b) $ab = ba$.
   (c) $\mathrm{ord}(a) = \mathrm{ord}(b)$.
   (d) $ab^{-1} \in H$.

15. Let H_1 and H_2 be two distinct subgroups of a finite abelian group G such that $H_1 H_2 = G$. Which of the following is necessarily true?

   (a) $|G| \leq |H_1| + |H_2|$.
   (b) $|G/H_1 \cap H_2| = |G/H_1||G/H_2|$.
   (c) $|G| = |H_1||H_2|$.
   (d) None of the above.

16. How many Sylow 2-subgroups are there in S_4 (symmetric groups on 4 letters)?

   (a) 1.
   (b) 2.
   (c) 3.
   (d) 4.

17. Let R be a commutative ring with two non-zero ideals I and J such that $I \cap J = (0)$. Which of the following is necessarily true?

   (a) R is not a domain.
   (b) $R = I + J$.
   (c) R is finite.
   (d) I and J are prime ideals.

18. Which of the following is necessarily true for the ring $R = \mathbb{Z}[\sqrt{2}]$?

   (a) R is an integral domain but is not a principal ideal domain.
   (b) R is principal ideal domain (PID) but is not an Euclidean domain.
   (c) R is an Euclidean domain but is not a field.
   (d) R is a field.

19. Consider the ideal $I = (x^2 - 2, y^2 + 1, z)$ in the ring $R = \mathbb{Q}[x, y, z]$. Which of the following is necessarily true?

(a) $R = I$.

(b) I is a maximal ideal.

(c) I is a prime ideal but not a maximal ideal.

(d) I is not a prime ideal.

2 Answers

1. D
2. C
3. A
4. D
5. C
6. B
7. D
8. B
9. C
10. D
11. A
12. B
13. C
14. D
15. B
16. C
17. A
18. C
19. B