PMB
2019

1. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying

$$\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = 0.$$

Show that f is a bounded function on $\mathbb{R}$ and attains a maximum or a minimum. Give an example to show that it attains a maximum but not a minimum.

2. Let $g : [0,1] \to \mathbb{R}$ be a continuous function such that $g(1) = 0$. Show that

$$\sup_{x\in[0,1]} |x^n g(x)| \to 0 \text{ as } n \to \infty.$$

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a twice continuously differentiable function and suppose $f(0) = f'(0) = 0$. If $|f''(x)| \leq 1$ for all $x \in \mathbb{R}$, then prove that $|f(x)| \leq 1/2$ for all $x \in [-1,1]$.

4. Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is a function defined by

$$f(x,y) = \begin{cases} \frac{x^2y^3}{x^4+y^6}, & \text{if } x \neq 0,\ y \in \mathbb{R}, \\ 0, & \text{if } x = 0,\ y \in \mathbb{R}. \end{cases}$$

   (a) Find all $(a,b) \in \mathbb{R}^2\backslash\{(0,0)\}$ such that f has a nonzero directional derivative at $(0,0)$ with respect to the direction (a,b).

   (b) Is f continuous at $(0,0)$? Justify your answer.

5. Let C be a subset of a compact metric space (X,d). Assume that for every continuous function $h : X \to \mathbb{R}$, the restriction of h to C attains a maximum on C. Prove that C is compact.

Please turn over

6. Let G be a non-abelian group of order pq, where $p < q$ are primes.

   (a) How many elements of G have order q?

   (b) How many elements of G have order p?

7. Prove or disprove the following statement: The ring $\mathbb{Q}[X]/(X^4 - 1)$ is isomorphic to a product of fields.

8. Let M be a symmetric matrix with real entries such that $M^k = 0$ for some $k \in \mathbb{N}$. Show that $M = 0$.

9. Suppose A and B are two $n \times n$ matrices with real entries such that the sum of their ranks is strictly less than n. Show that there exists a nonzero column vector $\mathbf{x} \in \mathbb{R}^n$ such that $A\mathbf{x} = B\mathbf{x} = \mathbf{0}$.

10. Suppose there are n persons in a party. Every pair of persons meet each other with probability $p \in (0, 1)$ independently of the other pairs. Let $N(i)$ be the number of people the i^{th} person meets in the party. For all $i, j \in \{1, 2, \ldots, n\}$ with $i \neq j$ and for all $k, l \in \{1, 2, \ldots, n-2\}$, show that

$$P\left[N(i) = k, N(j) = l\right] = \binom{n-2}{k-1}\binom{n-2}{l-1}p^{k+l-1}(1-p)^{2n-k-l-2} + \binom{n-2}{k}\binom{n-2}{l}p^{k+l}(1-p)^{2n-k-l-3}.$$