Putnam Problems Discussion

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Putnam problems discussion session 2,
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1) Let n be a positive integer, and let $f_n(z) = n + (n-1)z + (n-2)z^2 + \cdots + z^{n-1}$. Prove that f_n has no roots in the closed unit disk $\{z \in \mathbb{C} : |z| \leq 1\}$. (Putnam 2018- B2)

2) Find the smallest positive integer j such that for every polynomial $p(x)$ with integer coefficients and for every integer k, the integer

$$p^{(j)}(k) = \left.\frac{d^j}{dx^j}p(x)\right|_{x=k}$$

(the j-th derivative of $p(x)$ at k) is divisible by 2016. (Putnam 2016- A1)

3) Let $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ be sequences of positive real numbers such that $a_1 = b_1 = 1$ and $b_n = b_{n-1}a_n - 2$ for $n = 2, 3, \ldots$. Assume that the sequence (b_j) is bounded. Prove that

$$S = \sum_{n=1}^{\infty} \frac{1}{a_1 \cdots a_n}$$

converges, and evaluate S. (Putnam 2011 -A2)

4) Let $f : [0, \infty) \to \mathbb{R}$ be a strictly decreasing continuous function such that $\lim_{x\to\infty} f(x) = 0$. Prove that $\int_0^\infty \frac{f(x) - f(x+1)}{f(x)}\,dx$ diverges. (Putnam 2010 - A6)

5) Is there a strictly increasing function $f : \mathbb{R} \to \mathbb{R}$ such that $f'(x) = f(f(x))$ for all x? (HW) (Putnam 2010 - B5)

2) Find the smallest positive integer j such that for every polynomial $p(x)$ with integer coefficients and for every integer k, the integer

$$p^{(j)}(k) = \frac{d^j}{dx^j} p(x) \Big|_{x=k}$$

(the j-th derivative of $p(x)$ at k) is divisible by 2016.

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0.$$

$$p^{(1)}(k) = \sum_{r=0}^{n} r a_r k^{r-1}$$

$$p^{(j)}(k) = \sum_{r=j}^{n} \overbrace{r(r-1)(r-2)\dots(r-j+1)}^{\text{product of } j \text{ consecutive integer.}} a_r k^{r-j}$$

We can take $j!$ common from each summand as product of j conse. factor in divisible by $j!$

$$= j! \left\{ b_n a_n \cdot k^{n-j} + \dots + b_j a_j \right\}$$

$b_i \in \mathbb{Z}$, $i = 1, 2, \dots, j$ $\quad b_j = 1$.

Target → finding minm j

$\ni 2016 \mid j!$

$2016 = 2^5 \times 3^2 \times 7 \qquad j \geq 7.$

$2016 \mid 8! \qquad j_{min} = 8.$ (ans).

1) Let n be a positive integer, and let
$f_n(z) = n + (n-1)z + (n-2)z^2 + \cdots + z^{n-1}$.
Prove that f_n has no roots in the closed unit disk
$\{z \in \mathbb{C} : |z| \leq 1\}$.

$$f_n(z) = n + (n-1)z + \cdots + z^{n-1}$$
$$z \cdot f_n(z) = \qquad nz + \cdots + 2 \cdot z^{n-1} + z^n$$

$$(z-1) f_n(z) = z^n + z^{n-1} + \cdots + z - n.$$

$$\Rightarrow (z-1) f_n(z) + n = z^n + z^{n-1} + \cdots + z.$$

$$\Rightarrow \left| (z-1) f_n(z) + n \right| = \left| \sum_1^n z^k \right| \leq \sum_1^n |z|^k \longrightarrow ①.$$

assume it has a root $z_0 \ni |z_0| \leq 1$.

$f_n(z_0) = 0$. $\Rightarrow |z_0|^k \leq 1$. $k = 1, \ldots, n$

from ①,

$$n = \left| \sum_1^n z_0^k \right| \leq \sum_1^n |z_0|^k \leq \sum_1^n 1 = n.$$

- Equality occur.

So, $|z_0| = 1$. $\Rightarrow |z_0|^k = 1$, $k = 1, 2, \ldots, n$.

$\Rightarrow z_0, z_0^2, \ldots, z_0^n$ are concyclic. $\rightarrow$ (a)

Equality also occur in the triangular inea.

$\Rightarrow z_0, z_0^2, \ldots, z_0^n$ are collinear. $\rightarrow$ (b)

$f_n(1), f_n(-1) \neq 0.$

from (a) & (b) it's a $(\Rightarrow)(\Leftarrow)$.

5) Is there a strictly increasing function $f : \mathbb{R} \to \mathbb{R}$ such that $f'(x) = f(f(x))$ for all x?

suppose $\exists$ such f.

take $0 < a < b$.

$\Rightarrow \quad f(a) < f(b)$

from (S),

$0 < f(a) < f(b)$.

$$\int_a^b \frac{f'(x)}{f(f(x))} dx = \int_a^b 1.dx = b-a$$

$$b-a = \int_a^b \frac{f'(x)}{f(f(x))} dx.$$

$$= \int_{f(a)}^{f(b)} \frac{1}{f(y)} dy \leq \frac{1}{f(a)} \int_{f(a)}^{f(b)} dy = \frac{f(b)-f(a)}{f(a)}$$

$$\geq \frac{f(b)-f(a)}{f(b)}$$

f is one-one.

consider. $f(x) = y$

Suppose $\exists\, x_1 > 0$

$\ni f(x_1) < 0$

$\Rightarrow f^2(x_1) < f(0)$

$\Rightarrow f'(x_1) < f(0)$

$f'(x_1) > 0$

$\Rightarrow f(0) > 0$

$\Rightarrow$ since $x_1 > 0$

$\Rightarrow f(x_1) > f(0) > 0$

$(\Rightarrow)(\Leftarrow)$

So $f(x) > 0 \;\forall\, x > 0$.

$\hookrightarrow$ (S)

$$\Rightarrow \quad f(a) \leq \frac{f(b)-f(a)}{b-a} \leq f(b) \quad \forall\, a,b>0.$$

$$\Rightarrow \lim_{a\to b} f(a) \leq \lim_{a\to b} \frac{f(b)-f(a)}{b-a} \leq f(b)$$

$$\Rightarrow \quad f(b) \leq f'(b) \leq f(b)$$

$$\Rightarrow \quad f'(b) = f(b) \quad \forall\, b>0$$

$$\Rightarrow \quad f(b) = c.e^{b}. \quad \forall\, b>0. \qquad c \to \text{positive const.}$$

$$f(f(x)) = \underset{\overset{\|}{f(x)}}{f'(x)} \quad \forall\, x>0$$

$$\Rightarrow \quad f(f(x)) = f(x) \quad \forall\, x>0$$

as f is one-one

$$\Rightarrow \quad f(x) = x \quad \forall\, x>0$$

but we have $f(x) = c.e^{x} \ \forall\, x>0$

$(\Rightarrow\Leftarrow)$

$\nexists$ such f.

4) Let $f : [0, \infty) \to \mathbb{R}$ be a strictly decreasing continuous function such that $\lim_{x\to\infty} f(x) = 0$. Prove that $\int_0^\infty \frac{f(x) - f(x+1)}{f(x)}\, dx$ diverges.

f is str. $\downarrow$.
lt $f(x) = 0$
$\Rightarrow f > 0 \ \forall\, x \geq 0.$

case-1

$$\frac{f(x+1)}{f(x)} \geq \frac{1}{2} \text{ for finitely many pnts.}$$

then $\exists\ \lambda \in \mathbb{R}^+ \ni$

$$\frac{f(x+1)}{f(x)} < \frac{1}{2} \quad \forall\ x \geq \lambda.$$

$$\Rightarrow \quad \frac{f(x) - f(x+1)}{f(x)} > \frac{1}{2} \quad \forall\ x \geq \lambda.$$

$$\Rightarrow \int_0^\infty \frac{f(x) - f(x+1)}{f(x)}\, dx. \geq \int_\lambda^\infty \frac{f(x) - f(x+1)}{f(x)}\, dx$$

$$> \int_\lambda^\infty \frac{1}{2}\, dx$$

$\Rightarrow$ diverges.

case-2

$$\frac{f(x+1)}{f(x)} \geq \frac{1}{2} \text{ for infinitely many pnts.}$$

say. $\{x_1, x_2, x_3, \ldots\}$ this seq^n is unbdd. so will get such

λ like case-1.

$$\lim_{n\to\infty} f(x_n) = 0. \qquad \frac{f(x_{n+1})}{f(x_n)} \geq \frac{1}{2}$$

$$\int_{x_n}^{\infty} \frac{f(x)-f(x+1)}{f(x)}\,dx$$

$$> \frac{1}{f(x_n)} \int_{x_n}^{\infty} \big(f(x)-f(x+1)\big)\,dx$$

$$= \frac{1}{f(x_n)} \int_{x_n}^{x_n+1} f(x)\,dx \geq \frac{f(x_n+1)}{f(x_n)} \geq \frac{1}{2} \ \forall n.$$

(since f is $\downarrow$)

$$\Rightarrow \int_0^{\infty} \frac{f(x)-f(x+1)}{f(x)}\,dx \text{ diverges.}$$