Putnam Problems Discussion

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Putnam problems discussion session 1 ,
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1) Let $a_0 = 1$, $a_1 = 2$, $a_n = 4a_{n-1} - a_{n-2}$. for $n \geq 2$.
Find an odd prime factor of a_{2015}. (Putnam -2015 A2)

2) Consider a horizontal strip of $N+2$ squares in which the first and the last square are black and the remaining N squares are all white. Choose a white square uniformly at random, choose one of its two neighbors with equal probability, and color tis neighboring square black if it is not already black. Repeat this process until all the remaining white squares have only black neighbors. Let $w(N)$ be the expected number of white squares remaining. Find

$$\lim_{N\to\infty} \frac{w(N)}{N}.$$

(Putnam - 2020 A4)

3) Let $Q_0(x) = 1$. $Q_1(x) = x$ &

$$Q_n(x) = \frac{(Q_{n-1}(x))^2 - 1}{Q_{n-2}(x)} \quad \forall\ n \geq 2.$$

(Putnam 2017 A2)

P.T. whenever n is a positive integer, $Q_n(x)$ is equal to a polynomial with integer coefficients.

4) For each real coefficient polynomial $f(x) = a_0 + a_1 x + \cdots + a_n x^n$, let

$$\Gamma(f(x)) = a_0^2 + a_1^2 + \cdots + a_m^2.$$

Let be given polynomial $P(x) = (x+1)(x+2)\ldots(x+2020)$. Prove that there exists at least 2019 pairwise distinct polynomials $Q_k(x)$ with $1 \leq k \leq 2^{2019}$ and each of it satisfies two following conditions:
i) $\deg Q_k(x) = 2020$.
ii) $\Gamma\left(Q_k(x)^n\right) = \Gamma\left(P(x)^n\right)$ for all positive initeger n.

(Vietnam NMO)

5) $n \in \mathbb{Z}^+$, let the numbers $c(n)$ be determined by the rules $c(1)=1$, $c(2n)=c(n)$ & $c(2n+1)=(-1)^n c(n)$.

Find. $\displaystyle\sum_{n=1}^{2013} c(n).c(n+2)$.

(Putnam 2013 – B1)

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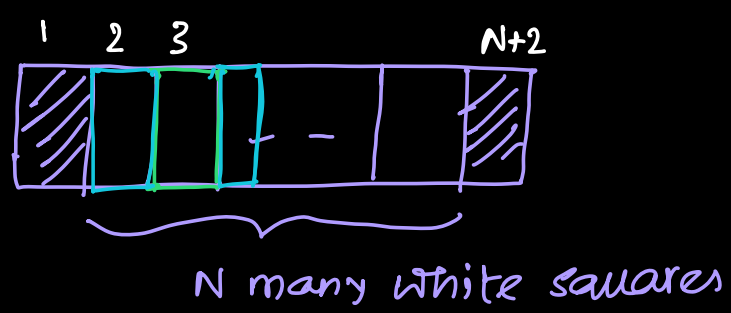
2) Consider a horizontal strip of $N+2$ squares in which the first and the last square are black and the remaining N squares are all white. Choose a white square uniformly at random, choose one of its two neighbors with equal probability, and color tis neighboring square black if it is not already black. Repeat this process until all the remaining white squares have only black neighbors. Let $w(N)$ be the expected number of white squares remaining. Find

$$\lim_{N\to\infty} \frac{w(N)}{N}.$$

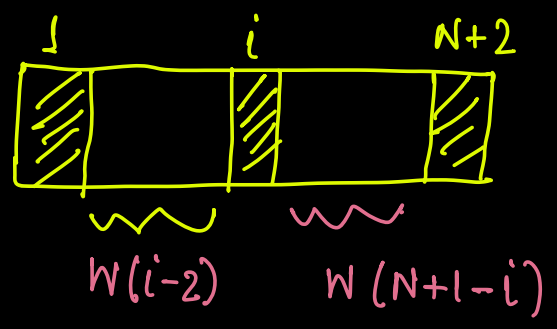
(Putnam – 2020 A4)



A = If I select a white square I can choose it's right neighbouring square to colour white $\to \{2,3,4,\ldots,N\}$.

B = - - - left . . $\to \{3,\ldots N+1\}$.

Suppose at first I have coloured black i^{th} square.



P(at first I have coloured black the ith square)

$i = 3, \dots, N$

$= P(\text{choosing } (i-1)^{th} \text{ square, } i^{th} \text{ sq is coloured black}) \times \frac{1}{2}$

$+ P(\text{choosing } (i+1)^{th} \text{ square, } i^{th} \text{ sq is col. black}) \times \frac{1}{2}$

$$= \frac{1}{2}\left(\frac{1}{N-1} + \frac{1}{N-1}\right) = \frac{1}{N-1}.$$

for $i=2$

$$W(N) = \sum_{i=3}^{N} \frac{W(i-2)+W(N+1-i)}{N-1} + \frac{W(0)+W(N-1)}{2(N-1)}$$

$$+ \frac{W(N-1)+W(0)}{2(N-1)} \rightarrow \text{for } i = N+1$$

$$= \sum_{i=2}^{N} \frac{W(i-2)+W(N+1-i)}{N-1}$$

$$(N-1)\,W(N) = \sum_{i=2}^{N} \left[W(i-2) + W(N+1-i) \right]$$

$$= 2\left(W(1) + \cdots + W(N-2) \right) + W(N-1)$$

$$(N-2)\,W(N-1) = 2\left(W(0) + \cdots + W(N-3) \right) + W(N-2)$$

$$(N-1)W(N) - (N-2)W(N-1)$$

$$= W(N-2) + W(N-1)$$

$$\Rightarrow \boxed{W(N) = W(N-1) + \frac{1}{N-1} W(N-2)}$$

$W(0) = 0, \quad W(1) = 1.$

claim: $W(N) = (N+1) \sum_{k=0}^{N+1} \frac{(-1)^k}{k!}$

$N = 0, 1 \rightarrow$ true.

let true for $N = m-1$.

$$W(m) = W(m-1) + \frac{1}{m-1} W(m-2)$$

$$= m\sum_{k=0}^{m} \frac{(-1)^k}{k!} + \frac{1}{m-1} \times m-1 \sum_{k=0}^{m-1} \frac{(-1)^k}{k!}$$

$$= (m+1)\sum_{k=0}^{m-1} \frac{(-1)^k}{k!} + \underset{(m+1-1)}{m} \frac{(-1)^m}{m!}$$

$$= (m+1)\sum_{k=0}^{m-1} \frac{(-1)^k}{k!} + (m+1)\frac{(-1)^m}{m!} - \frac{(-1)^m}{m!}$$

$$= (m+1)\sum_{k=0}^{m} \frac{(-1)^k}{k!} + \frac{(-1)^{m+1}}{(m+1)!}(m+1)$$

$$= (m+1)\sum_{k=0}^{m+1} \frac{(-1)^k}{k!}$$

- our claim is proved.

$$W(N) = (N+1)\sum_{k=0}^{N+1} \frac{(-1)^k}{k!}$$

$$\frac{W(N)}{N+1} = \sum_{k=0}^{N+1} \frac{(-1)^k}{k!} \longrightarrow e^{-1}. \quad (\text{as } N \to \infty)$$

4) For each real coefficient polynomial $f(x) = a_0 + a_1 x + \cdots + a_n x^n$, let

$$\Gamma(f(x)) = a_0^2 + a_1^2 + \cdots + a_n^2.$$

Let be given polynomial $P(x) = (x+1)(x+2)\ldots(x+2020)$. Prove that there exists at least 2019 pairwise distinct polynomials $Q_k(x)$ with $1 \le k \le 2^{2019}$ and each of it satisfies two following conditions:

i) $\deg Q_k(x) = 2020$.

ii) $\Gamma\left(Q_k(x)^n\right) = \Gamma\left(P(x)^n\right)$ for all positive initeger n.

(Vietnam NMO)

$$\Gamma(f(x)) = \sum_{i=0}^{n} a_i^2 = [x^n]\left\{(a_0 + a_1 x + \cdots + a_n x^n) \times \underbrace{(a_n + a_{n-1}x + \cdots + a_0 x^n)}_{f\left(\frac{1}{x}\right) \times x^n}\right\}$$

$$= [x^n]\left\{f(x) \times f\left(\frac{1}{x}\right) \times x^n\right\}$$

$$P(x)^n = \prod_{i=1}^{2020} (x+i)^n$$

$$\Gamma(P(x)^n) = [x^{2020}]\left\{\prod_{i=1}^{2020} (x+i)^n \times x^{2020n} \times \prod_{i=1}^{2020} \left(\frac{1}{x}+i\right)^n\right\}$$

$$= [x^n]\left\{\prod_{i=1}^{2020} (x+i)^n \times \prod_{i=1}^{2020} (ix+1)^n\right\}$$

$$Q_k(x)^n = \prod_{i=1}^{k-1} (x+i)^n \times \prod_{i=k+1}^{2020} (x+i)^n \times (kx+1)^n$$

$k = 2, \ldots, 2020$

$$Q_k\left(\frac{1}{x}\right)^n = \prod_{i=1}^{k-1} \left(\frac{1}{x}+i\right)^n \times \prod_{k+1}^{2020} \left(\frac{1}{x}+i\right)^n \times \left(\frac{k}{x}+1\right)^n$$

$$= \prod_{1}^{k-1} (ix+1)^n \times \prod_{k+1}^{2020} (ix+1)^n \times (x+k)^n \times \frac{1}{x^{2020n}}$$

$$\left[x^{2020}\right]\left\{Q_k(x) \times Q_k\left(\frac{1}{x}\right) x^{2020n}\right\} = \left[x^{2020}\right]\left\{P(x) \times P\left(\frac{1}{x}\right) \times x^{2020n}\right\}$$

$$\Rightarrow \Gamma\left(Q_k(x)^n\right) = \Gamma\left(P(x)^n\right).$$

$$\deg Q_k(x) = 2020.$$

$Q_2(x), Q_3(x), \ldots, Q_{2020}(x) \rightarrow$ pairwise distinct

— done.

1) Let $a_0 = 1$, $a_1 = 2$, $a_n = 4a_{n-1} - a_{n-2}$. for $n \geq 2$.
Find an odd prime factor of a_{2015}. (Putnam -2015 A2)

$$a_n = 4a_{n-1} - a_{n-2}$$

$$a_n - 4a_{n-1} + a_{n-2} = 0$$

$$a_n \equiv \lambda^n$$

$$\lambda^2 - 4\lambda + 1 = 0 \rightarrow \lambda = \frac{4 \pm \sqrt{16-4}}{2}$$

$$= \frac{4 \pm 2\sqrt{3}}{2} = 2 \pm \sqrt{3}$$

$$a_n = a(2+\sqrt{3})^n + b(2-\sqrt{3})^n.$$

$$a_0 = 1 \Rightarrow a + b = 1$$

$$a_n = a(2+\sqrt{3})^n + (1-a)(2-\sqrt{3})^n$$

$$a_1 = 2$$

$$a(2+\sqrt{3}) + (1-a)(2-\sqrt{3}) = 2$$

or, $a(\cancel{2}+\sqrt{3}-\cancel{2}+\sqrt{3}) = \sqrt{3}$

or, $a = \frac{1}{2}$

$$a_n = \frac{1}{2}\left\{(2+\sqrt{3})^n + (2-\sqrt{3})^n\right\}.$$

$$= \frac{1}{2}\left\{(2+\sqrt{3})^n + \frac{1}{(2+\sqrt{3})^n}\right\}$$

$n \to$ odd.

$$(2+\sqrt{3})^m + (2-\sqrt{3})^m \;\Big|\; (2+\sqrt{3})^n + (2-\sqrt{3})^n$$

$n = dm \quad d, n, m \to$ odd.

$d = 2k+1.$

$2+\sqrt{3} = a$

$2-\sqrt{3} = b.$

$a = 1/b$

$$a_m = a^{(2k+1)m} + b^{(2k+1)m}$$

$$= (a^m + b^m)\left(a^{2km} - b^m \cdot a^{m(2k-1)} + \cdots + b^{2km}\right\}$$

$(2k+1)$

$$= (a^m + b^m)\underbrace{\left\{a_{2km} - a_{(2k-1)m} + a_{(2k-2)m} - \cdots \pm 1\right\}}_{\in \mathbb{Z}.}$$

$a^m + b^m \mid a^n + b^n$ when $n = \text{odd} \times m.$

$\therefore a_5 \mid a_{2015}.$ $\quad a_5 = 362 = 2 \times 181.$

$a_m \to 181.$

3) Let $Q_0(x) = 1$. $Q_1(x) = x$ &

(Putnam 2017) A2

$$Q_n(x) = \frac{(Q_{n-1}(x))^2 - 1}{Q_{n-2}(x)} \quad \forall \; n \geq 2.$$

P.T. whenever n is a positive integer, $Q_n(x)$ is equal to a polynomial with integer coefficients.

$\rightarrow$ $Q_0(x) = 1$, $Q_1(x) = x$.

$$Q_n(x) \rightarrow \lambda_n.$$

$n=2$, $\lambda_2 = \frac{x^2-1}{1} = x^2 - 1$.

$n=3$, $\lambda_3 = \frac{(x^2-1)^2 - 1}{x} = x(x^2-2)$.

$n=4$, $\lambda_4 = \frac{x^2(x^2-2)^2 - 1}{x^2-1} = \frac{x^2(x^4-4x^2+4)-1}{x^2-1}$

$$= x^4 + x^2 + 1 - 4x^2$$

$$= x^4 - 3x^2 + 1$$

$$= x \cdot x(x^2-2) - (x^2-1).$$

claim: $\lambda_n = x\lambda_{n-1} - \lambda_{n-2}$. $\forall\, n \geq 3$.

$n = 3, 4 \to$ true.

let true till $n = k-1$.

$$\lambda_k = \frac{\lambda_{k-1}^2 - 1}{\lambda_{k-2}} = \frac{(x\lambda_{k-2} - \lambda_{k-3})^2 - 1}{\lambda_{k-2}}$$

$$= x^2\lambda_{k-2} - 2x\lambda_{k-3} + \frac{\lambda_{k-3}^2 - 1}{\lambda_{k-2}}$$

$$= x^2\lambda_{k-2} - 2x\lambda_{k-3} + \lambda_{k-4}$$

$$= x(x\lambda_{k-2} - \lambda_{k-3}) - (x\lambda_{k-3} - \lambda_{k-4})$$

$$= x\lambda_{k-1} - \lambda_{k-2}$$

— done.

$$Q_n(x) = x \cdot Q_{n-1}(x) - Q_{n-2}(x)$$

$Q_0(x)$, $Q_1(x)$, $Q_2(x) \in \mathbb{Z}[x]$

by induction, $Q_n(x) \in \mathbb{Z}[x]$.