Handwritten Notes on Number Theory & Combinatorics

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ABOUT MY COURSES ON THIS PLATFORM:

I will mostly teach problem-solving methods through problem sessions and a breif recap of the introductory topics.

WHAT I WILL TEACH IN THIS WORKSHOP:

I will mostly focus on NUMBER THEORY and COMBINATORICS at the level of 10+2 for ISI and CMI entrances. Even if you are appearing for other Entrances / Interviews this will boost your Problem Solving skills.

SYLLABUS & STRUCTURE:

The syllabus would be as per the Number Theory & Combinatorics sections mentioned in ISI BROCHURE, available online. The course will be divided into : →

1. Online Classes & Problem Sessions.
2. Model Test Papers.
3. Handwritten Notes & Problem Sheet.

[Some cheat-sheets / guides] will be latexed.

ABOUT THE LECTURES:

I would have some categorized sections. Try proving all the INTERESTING FACTS and go through the SOLVED EXAMPLES for understanding the techniques. And finally PROBLEM SOLVING APPROACH for facing an unknown problem.

WHAT TO EXPECT:

Several problem solving tools / techniques for approaching NT & Combinatorics problems through interesting examples.

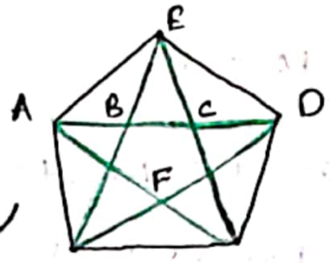
NUMBER THEORY
&
COMBINATORICS

INTRO TO LECTURE 1 (PART 1)

Lecture notes : Diganta B. B.Stat ISI Kolkata.
M.Stat. Currently pursuing.

PROBLEM: Look at a regular pentagon →

What is $\angle AFD$? What is $\angle ADF$?



Diagonals forming a pentagonal star.

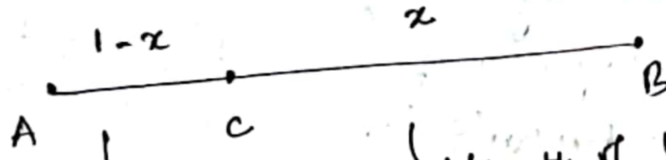
$\angle AFD = 108^\circ$ & $\angle ADF = 36^\circ$. What is the ratio? $\frac{AD}{AF}$.

$AF = AC$ from the regular pentagon. Use sine rule →

$$\frac{AD}{AF} = \frac{\sin 108^\circ}{\sin 36^\circ} = 2\cos 36^\circ = 2 \cdot \left[\frac{1+\sqrt{5}}{4}\right] = \alpha \text{ [Say]}.$$

Now, $\frac{AD}{AF} = \frac{AD}{AC} = \alpha$. What is $\frac{AC}{CD}$? [Check Digression.]

Digression: Draw a line segment $\overline{AB}$ of length 1 and divide it into 2 parts, $\overline{AC}$ & $\overline{CB}$. Divide this segment such that the ratio of the whole segment to the larger part is equal to the ratio of the larger part to the smaller.



→ we have produced: $\frac{1}{x} = \frac{x}{1-x} \Rightarrow x^2 = 1-x$.

↳ length of the larger portion

length of the smaller portion

$1/x = \frac{1+\sqrt{5}}{2} = \alpha$ [defined above] → This ratio is called Golden Ratio & the division of this line at point C is called the Golden Section.

$\frac{AC}{CD} = \alpha$ by Golden Section & $AB = CD \Rightarrow \frac{AC}{AB} = \frac{AB}{BC} = \alpha$.

INTERESTING FACT 1:

The segments BC, AB, AC and AD each is α times greater than the preceeding one.

Now, we know that we are dealing with Fibonacci Numbers right? Before we dive deeper and explore the techniques in Number Theory & combinatorics through Fibonacci, let us state some more interesting facts about Fibonacci.

INTERESTING FACT 2:

If we calculate ratio of two successive fibonacci numbers say f_{n+1} & f_n then $\lim_{n \to \infty} \frac{f_{n+1}}{f_n} = \frac{1+\sqrt{5}}{2}$.

INTERESTING FACT 3:

The sum of the numbers along a raising diagonal in a Pascal's Triangle is a Fibonacci Number.

Hint: $(n-1)^{th}$ diagonal has entries $\rightarrow$

${}^{0}C_{k-2}, {}^{1}C_{n-3}, {}^{2}C_{n-5}$ where

${}^{i}C_{j} = k! / i!(k-i)!$ i $\rightarrow$ column & k $\rightarrow$ row.

Raising diagonals of a Pascal's Δ.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

INTERESTING FACT 4:

If we denote i^{th} Fibonacci by f_i then $\rightarrow$ $f_{2n} = f_{n+1}^2 - f_{n-1}^2$.

Hint: Haha! Check the next result; if this seems intriguing.

INTERESTING FACT 5:

Using same notation, $f_{n+m} = f_{n-1} f_m + f_n f_{m+1}$.

Hint: Use Induction.

INTERESTING FACT 6:

The sum of squares of first - n - fibonacci numbers:

$$f_1^2 + f_2^2 + \dots + f_n^2 = f_n f_{n+1}$$

INTERESTING FACT 7:

The sum of fibonacci numbers with alternating signs:

$$f_1 - f_2 + f_3 - f_4 + \dots + (-1)^{n+1} f_n = (-1)^{n+1} f_{n-1} + 1$$

INTERESTING FACT 8:

Sum of even terms of fibonacci sequence →

$$f_2 + f_4 + f_6 + \dots + f_{2n} = f_{2n+2} - 1$$

INTERESTING FACT 9:

Sum of odd terms of fibonacci sequence →

$$f_1 + f_3 + f_5 + \dots + f_{2n-1} = f_{2n}$$

Order to prove easily.

INTERESTING FACT 10: (Prove $10 \to 9 \to 8 \to 7 \to 6 \to 5 \to 4$)

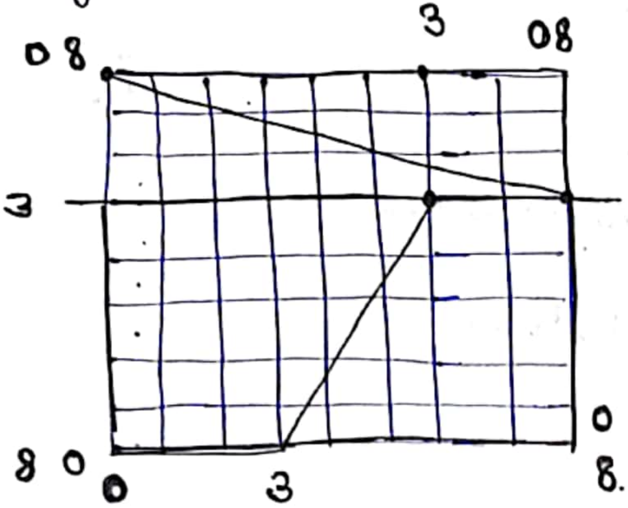
The sum of first 'n' fibonacci numbers can be expressed as →

$$f_1 + \dots + f_n = f_{n+2} - 1$$

An interesting paradox to be discussed later →

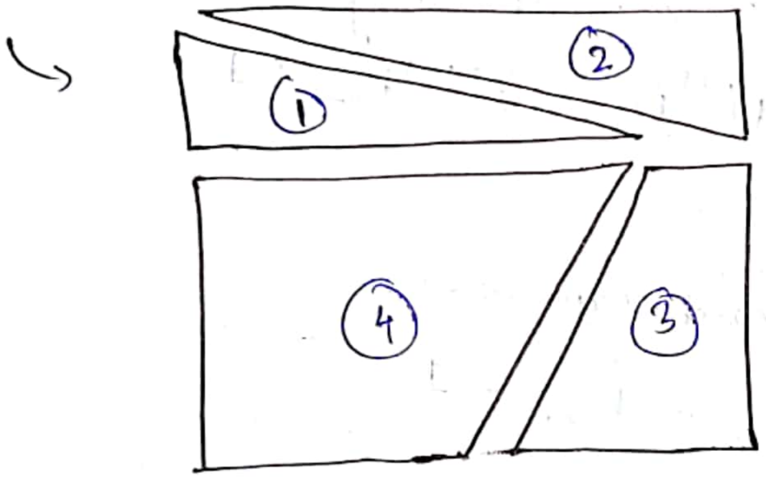
• We shall show that $\boxed{64=65}$!! (Video link)

You might have seen this as a magic trick! Do you understand now why this works?

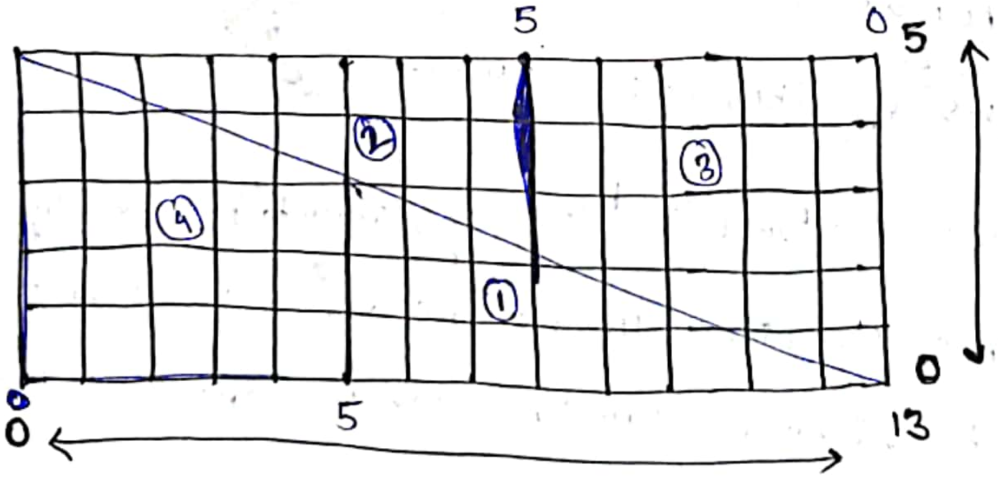


Assume this is a 8x8 regular grid.

If we cut along the black lines into 4 parts ⟶



Something like this. Rearranging these 4 parts into a $\boxed{13 \times 5}$ rectangle, we now have 65 squares.



You might observe there is a small gap which is along the diagonal in this 13×5 rectangle which accounts for the missing square. Use a larger Fibonacci no. to denote the side of the square and check!

The n-th term of the Fibonacci Sequence is →

$$f_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}} \rightarrow \text{Binet's formula.}$$

(How??)

① TRICKBOOK [Generating functions]

What is a generating function precisely?

⇒ For now let us say a function in which the coefficients of the power series gives answer to a counting problem.

Let us work out the Binet formula using Generating functions & after we finish modular arithmetic we will dwelve deep into this trick | way to approach a counting problem. We will formally define it in class and see more examples.

WORKED OUT EXAMPLE [Generating functions]

$f_n = f_{n-1} + f_{n-2}$ → fibonacci Sequence.

$$f_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}} \rightarrow \text{Binet's formula.}$$

Define $f(x) = f_0 + f_1 x + f_2 x^2 + f_3 x^3 + \dots\dots$

$$-x \cdot f(x) = f_0(-x) + f_1 x(-x) + \dots\dots$$

$$= -f_0 x + f_1 \cdot (-x^2) + f_2 \cdot (-x^3) \dots$$

$$-x^2 \cdot f(x) = -f_0 x^2 - f_1 x^2 - \dots\dots$$

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$$f(x) = f_0 + f_1 x + f_2 x^2 + f_3 x^3 + \dots$$
$$-x \cdot f(x) = -f_0 x - f_1 x^2 - f_2 x^3 \dots$$
$$-x^2 \cdot f(x) = -f_0 x^2 - f_1 x^3 - f_2 x^4 \dots$$

Adding them up like we did for AGP.

$$f(x) - x \cdot f(x) - x^2 \cdot f(x) = f_0 + (f_1 - f_0)x = f_0 = 1.$$

$$\& \quad f(x) = \frac{1}{1 - x - x^2} = \frac{-1}{x^2 + x - 1}$$

What are its roots?

$$x = \frac{-1+\sqrt{5}}{2} \qquad x = \frac{-1-\sqrt{5}}{2}$$

(I) Find the roots for partial fractions.

(II) Use method of partial fractions to break $f(x)$ even further

$$f(x) = \frac{-1}{x^2+x-1} = \frac{-1}{\left(x - \frac{-1+\sqrt{5}}{2}\right)\left(x - \frac{-1-\sqrt{5}}{2}\right)} = \frac{A}{\left(x + \frac{1-\sqrt{5}}{2}\right)} + \frac{B}{\left(x + \frac{1+\sqrt{5}}{2}\right)}$$

Solve for A, B $\rightarrow$ $A = -1/\sqrt{5}$ $\quad B = 1/\sqrt{5}$.

$$\Rightarrow f(x) = \frac{-1}{x^2+x-1} = \frac{-1/\sqrt{5}}{x + \frac{1-\sqrt{5}}{2}} + \frac{1/\sqrt{5}}{x + \frac{1+\sqrt{5}}{2}}$$

(III) Use the notation for negative binomial $\rightarrow \binom{-1}{k} = \frac{(-1)(-2)\dots(-1+1-k)}{k!} = (-1)^k$.

$$f(x) = \frac{-1/\sqrt{5}}{x + \frac{1-\sqrt{5}}{2}} + \frac{1/\sqrt{5}}{x + \frac{1+\sqrt{5}}{2}} = \frac{-1}{\sqrt{5}} \sum_{k=0}^{\infty} \binom{-1}{k} x^k \left(\frac{1-\sqrt{5}}{2}\right)^{-1-k} + \frac{1}{\sqrt{5}} \sum_{k=0}^{\infty} \binom{-1}{k} x^k \left(\frac{1+\sqrt{5}}{2}\right)^{-1-k}$$

$$= \frac{1}{\sqrt{5}} \sum_{k=0}^{\infty} \left[(-1)^{k+1} \left(\frac{1-\sqrt{5}}{2}\right)^{-1-k} + (-1)^k \left(\frac{1+\sqrt{5}}{2}\right)^{-1-k} \right] x^k = \frac{1}{\sqrt{5}} \sum_{k=0}^{\infty} \left[\left(\frac{-2}{1-\sqrt{5}}\right)^{k+1} - \left(\frac{2}{1+\sqrt{5}}\right)^{k+1} \right] x^k.$$

$\rightarrow$ This generating function was for the recursion $f_n = f_{n-1} + f_{n-2}$.

$$k^{th}\text{ coefficient of } x^k = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^{k+1}$$

Same as <u>Binet</u>!