Sequences Discussion

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Monotone Sequences

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⊙ THEOREM : (MONOTONE CONVERGENCE THEOREM / MCT)

(A) If $\{x_n\}$ is BOUNDED ABOVE and INCREASING, then $\{x_n\}$ converges.

(B) If $\{x_n\}$ is BOUNDED BELOW and DECREASING, then $\{x_n\}$ converges.

Q) ① B.STAT - 2016 : Problem # 8

$\{a_n\}$ is a sequence, s.t. $a_{n+1} = \dfrac{3a_n}{2+a_n}$ $\forall\, n \geq 1$

(1) If $0 < a_1 < 1$, then P.T. $a_n \uparrow$, and $a_n \to 1$.

(2) If $a_1 > 1$, then P.T. $a_n \downarrow$, and $a_n \to 1$.

Solution : (1) Suppose, $0 < a_1 < 1$. We claim that $a_n < 1$ for all $n \in \mathbb{N}$.

Suppose, $a_n < 1$ for some $n = k$, i.e. $a_k < 1$

$$\Rightarrow \frac{1}{a_k} > 1$$

$$\frac{1}{a_{k+1}} = \frac{2}{3}\cdot\frac{1}{a_k} + \frac{1}{3} > \frac{2}{3}\times 1 + \frac{1}{3} = 1 \Rightarrow a_{k+1} < 1$$

Thus, $a_k < 1 \Rightarrow a_{k+1} < 1$. But, $a_1 < 1$

So, by Induction, $\boxed{a_n < 1}$ for all $n \in \mathbb{N}$.

We have : $a_{n+1} = \dfrac{3a_n}{2+a_n} > \dfrac{3a_n}{2+1} = a_n$, $\forall n \in \mathbb{N}$

Thus, $\{a_n\}$ is strictly increasing.

So, $\{a_n\}$ is bounded above by 1, and $\{a_n\}$ is also strictly increasing. Therefore, $\{a_n\}$ converges. [THEOREM - 1]

Suppose, $a_n \to a$, i.e. $\lim_{n\to\infty} a_n = a$. So,

$$a = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{3a_n}{2+a_n} = \frac{3a}{2+a}$$

$\Rightarrow \quad 2a + a^2 = 3a$

$\Rightarrow \quad a^2 = a \quad \Rightarrow \boxed{a = 1} \quad [\because a \neq 0]$

i.e. $a_n \to 1$

$a_n > a_1$

$\Rightarrow \lim_{n\to\infty} a_n \geq a_1$

$\Rightarrow a \geq a_1 > 0$

$\Rightarrow a > 0$ ▨

Q 2) Suppose, $\{x_n\}$ is a sequence, s.t. $\forall n \in \mathbb{N}$:

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{\lambda}{x_n}\right) \quad \text{for some fixed } \lambda > 0.$$

If $x_1 \geq \sqrt{\lambda}$, then P.T. $\{x_n\}$ converges, and find its limit.

Solution : [Lemma] If $a, b \geq 0$, then $a+b \geq 2\sqrt{ab}$.

find its limit.

Solution :

$\boxed{\text{LEMMA}}$ If $a, b \geq 0$, then $a+b \geq 2\sqrt{ab}$.

Proof : $(\sqrt{a} - \sqrt{b})^2 \geq 0$

$\Rightarrow \quad a - 2\sqrt{ab} + b \geq 0$

$\Rightarrow \quad a + b \geq 2\sqrt{ab}$ ▨

Clearly, $x_n > 0$ for all $n \in \mathbb{N}$.

Now, $$x_{n+1} = \frac{1}{2}\left(x_n + \frac{\lambda}{x_n}\right) \geq \frac{1}{2} \cdot 2\sqrt{x_n \cdot \frac{\lambda}{x_n}} = \sqrt{\lambda}, \quad \forall n \in \mathbb{N}$$

So, $\{x_n\}$ is BOUNDED BELOW by $\sqrt{\lambda}$.

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{\lambda}{x_n}\right) \leq \frac{1}{2}\left(x_n + x_n\right) = x_n$$

$\left[\because x_n \geq \sqrt{\lambda} \Rightarrow x_n^2 \geq \lambda \right.$
$\left. \Rightarrow x_n \geq \frac{\lambda}{x_n}\right]$

Thus, $\{x_n\}$ is decreasing, and bounded below.

Hence, $\{x_n\}$ converges, say $x_n \longrightarrow A$.

So, $$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \frac{1}{2}\left(x_n + \frac{\lambda}{x_n}\right)$$

$$\Rightarrow \quad A = \frac{1}{2}\left(A + \frac{\lambda}{A}\right)$$

$$\Rightarrow \quad A^2 = \lambda$$

$$\Rightarrow \quad A = \pm\sqrt{\lambda} \quad \Rightarrow \quad A = \sqrt{\lambda}$$

$[\because\ x_n > 0$ for all $n \in \mathbb{N}$

$\Rightarrow \lim\limits_{n\to\infty} x_n \geq 0$

$\Rightarrow A \geq 0 > -\sqrt{\lambda}$

$\Rightarrow A > -\sqrt{\lambda} \Rightarrow A \neq -\sqrt{\lambda}]$

Thus, $\lim\limits_{n\to\infty} x_n = \sqrt{\lambda}$. $\blacksquare$

H/W

$\boxed{\text{Q-1}}$ Suppose $\{Z_n\}$ is a sequence satisfying the relation :

$$Z_{n+1} = \sqrt{\frac{ab^2 + Z_n^2}{a+1}}$$, for all $n \in \mathbb{N}$. It is given that $0 < a < b$, and $Z_1 = a$.

P.T. $\{Z_n\}$ converges, and find $\lim\limits_{n\to\infty} Z_n$.

$\boxed{\text{Q-2}}$ (A) Let $\{x_n\}$ be any sequence of real numbers.

P.T. $\{x_n\}$ has a monotone subsequence.

VERY IMPORTANT

(B) Suppose $\{x_n\}$ is a bounded sequence.

P.T. $\{x_n\}$ has a convergent subsequence.

$\boxed{\text{Q-3}}$ B. STAT - 2019 : Problem #5

For all $n \in \mathbb{N}$, define :

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$$A_n = \sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots}}}}$$

[There are n many square roots in A_n]

(A) For all $n \geqslant 2$, P.T. $A_n = 2 \sin\left(\frac{\pi}{2^{n+1}}\right)$

(B) Evaluate : $\lim\limits_{n \to \infty} 2^n A_n$

Q-4 B. STAT - 2017 : Problem # 5

Let $g : \mathbb{N} \to \mathbb{N}$, such that $g(n)$ is the product of the digits of n.

(A) P.T. $g(n) \leqslant n$, for all $n \in \mathbb{N}$.

(B) Find all solutions to the equation :

$$n^2 - 12n + 36 = g(n)$$

Q-5 B. STAT - 2017 : Problem # 1

The sequence $\{a_n\}$ is defined as $a_n = \tan(n\theta)$ for all $n \in \mathbb{N}$, where $\tan(\theta) = 2$.

P.T. for all $n \in \mathbb{N}$, a_n is a rational number which can be written with an odd denominator.