Solutions to the 2012 CMI BSc Entrance Examination

Part A: 5 problems $\times$ 6 marks. **Part B: 7 out of 9 problems $\times$ 10 marks.**

A1. Find the number of real solutions to the equation $x = 99\sin(\pi x)$.

The number of solutions is the number of times the line $y = \frac{x}{99}$ meets the graph of $y = \sin(\pi x)$. This can occur only for $x \in [-99, 99]$ because $\sin(\pi x)$ has range $[-1, 1]$. Also $\sin(\pi x)$ is periodic with period 2. For $x \geq 0$, the two graphs meet twice in each cycle of $\sin(\pi x)$, both intersections occurring in the first half of the cycle. There are 50 such half-cycles from $x = 0$ to $x = 99$, over intervals $[0, 1], [2, 3], \ldots, [98, 99]$. So there are 100 non-negative solutions. Similarly there are 100 solutions ≤ 0 because both graphs are odd. Since $x = 0$ is counted twice, the total number of solutions is $100 + 100 - 1 = 199$.

A2. A differentiable function $f : \mathbb{R} \to \mathbb{R}$ satisfies $f(1) = 2, f(2) = 3$ and $f(3) = 1$. Show that $f'(x) = 0$ for some x.

Since f is differentiable, it is continuous. By the intermediate value theorem, there is $a \in (2, 3)$ with $f(a) = 2 = f(1)$. Now by Rolle's theorem there is $x \in (1, a)$ with $f'(x) = 0$. –OR– The continuous function f over the closed interval $[1, 2]$ must attain its absolute maximum, which cannot be at either endpoint (since $f(2) > f(1)$ and $f(2) > f(3)$). So the maximum must be at an interior point x and then $f'(x) = 0$. –OR– By the mean value theorem, $f'(y) = 1 > 0$ for some $y \in (1, 2)$ and $f'(z) = -2 < 0$ for some $z \in (2, 3)$. So $f'(x) = 0$ for some $x \in (y, z)$ since for a differentiable f, the function f' satisfies the intermediate value property by Darboux's theorem. (*This is important to say because f' need not be continuous.*)

A3. Show that $\frac{\ln(12)}{\ln(18)}$ is irrational.

$\frac{\ln(12)}{\ln(18)} = \log_{18}(12)$. Suppose this is rational, say $= \frac{a}{b}$ where a, b are integers with $b \neq 0$. Then $18^{\frac{a}{b}} = 12$, so $18^a = 12^b$. By factoring into primes this gives $3^{2a}2^a = 3^b 2^{2b}$, which by unique factorization can happen only if $2a = b$ and $a = 2b$. But this gives $a = b = 0$, a contradiction. (Alternatively and similarly, prove that $r = \frac{\ln(2)}{\ln(3)}$ is irrational and show that rationality of $\frac{\ln(12)}{\ln(18)} = \frac{\ln 3 + 2\ln 2}{2\ln 3 + \ln a} = \frac{1+2r}{2+r}$ would force r to be rational as well.)

A4. Show that

$$\lim_{x \to \infty} \frac{x^{100}\ln(x)}{e^x \tan^{-1}(\frac{\pi}{3} + \sin x)} = 0\,.$$

There is a positive constant c such that $\tan^{-1}(\frac{\pi}{3} + \sin x) > c$ for any x, e.g. $c = \tan^{-1}(0.04)$ will work since $\pi > 3.12$, $\sin(x) \geq -1$ and $\tan^{-1}$ is an increasing function. Moreover $\ln(x) < x$ for $x > 0$. So the given ratio is sandwiched between 0 and x^{101}/ce^x. Now use L'Hospital's rule repeatedly.

A5. a) n identical chocolates are to be distributed among the k students in Tinku's class. Find the probability that Tinku gets at least one chocolate, assuming that the n chocolates are handed out one by one in n independent steps. At each step, one chocolate is given to a randomly chosen student, with each student having equal chance to receive it.

P(Tinku gets at least one chocolate) $= 1 -$ P(Tinku gets none) $= 1 - (1 - \frac{1}{k})^n$, because in each of the independent steps the probability of Tinku not getting a chocolate is $1 - \frac{1}{k}$.

b) Solve the same problem assuming instead that all distributions are equally likely. You are given that the number of such distributions is $\binom{n+k-1}{k-1}$. (Here all chocolates are considered interchangeable but students are considered different.)

There are $\binom{(n-1)+k-1}{k-1}$ distributions in which Tinku gets at least a chocolate: give Tinku a chocolate and then use the given formula to find number of distributions of the remaining $n-1$ chocolates among k students. So the answer is $\binom{(n-1)+k-1}{k-1}/\binom{n+k-1}{k-1} = \frac{n}{n+k-1}$. –OR– The number of distributions in which Tinku gets no chocolate = number of distributions of n chocolates among the remaining $k-1$ students $= \binom{n+k-2}{k-2}$. So the desired probability is $1 - \binom{n+k-2}{k-2}/\binom{n+k-1}{k-1} = \frac{n}{n+k-1}$.

B1. a) Find a polynomial $p(x)$ with *real* coefficients such that $p(\sqrt{2}+i) = 0$.

Non-real roots of a polynomial with real coefficients occur in conjugate pairs. $p(x) = (x - (\sqrt{2}+i))(x - (\sqrt{2}-i)) = x^2 - 2\sqrt{2}x + 3$ works.

b) Find a polynomial $q(x)$ with *rational* coefficients and having the smallest possible degree such that $q(\sqrt{2}+i) = 0$. Show that any other polynomial with rational coefficients and having $\sqrt{2}+i$ as a root has $q(x)$ as a factor.

$\sqrt{2}+i$ satisfies $x^2 - 2\sqrt{2}x + 3 = 0$, i.e., $x^2 + 3 = 2\sqrt{2}x$ and so satisfies $(x^2+3)^2 = 8x^2$. So $q(x) = (x^2+3)^2 - 8x^2$ works. A cubic with rational coefficients will not work because, after dividing by the necessarily rational leading coefficient, it must be of the form $(x^2 - 2\sqrt{2}x + 3)(x - r)$. This forces the coefficients $-3r$ and $-2\sqrt{2} - r$ to be both rational, which is impossible.

Let $f(x)$ be a polynomial with rational coefficients such that $f(\sqrt{2}+i) = 0$. Divide $f(x)$ by $q(x)$ using long division to get quotient $a(x)$ and remainder $b(x)$, both polynomials with rational coefficients. Using $f(\sqrt{2}+i) = 0$ and $q(\sqrt{2}+i) = 0$ in the equation $f(x) = q(x)a(x) + b(x)$ gives $b(\sqrt{2}+i) = 0$. Now if the remainder $b(x)$ is a nonzero polynomial, then it would have rational coefficients, degree less than 4 and $\sqrt{2}+i$ as a root. But we just proved that this is impossible. Hence $b(x) = 0$, i.e., $f(x)$ is a multiple of $q(x)$.

B2. a) Let E, F, G and H respectively be the midpoints of the sides AB, BC, CD and DA of a convex quadrilateral ABCD. Show that EFGH is a parallelogram whose area is half that of ABCD.

Consider the diagonals AC and BD. By the basic proportionality theorem in triangle ABC, we get that EF and AC are parallel and AC = 2 EF. Moreover, ABC and EBF are similar. Using triangles ADC and HDG, we similarly get that AC is parallel to HG, AC = 2 HG. Thus EF and HG are parallel. Likewise FG and EH are parallel (both parallel to BD), so EFGH is a parallelogram. Also by similarity, Area(ABC) = 4 Area(EBF), Area(ADC) = 4 Area(HDG), Area(BAD) = 4 Area(EAH) and Area(BCD) = 4 Area(FCG). (*Note.* So far convexity of ABCD is unnecessary. But the next steps need it, draw pictures and see.)

Area(EFGH) = Area(ABCD) – [Area(EBF) + Area(FCG) + Area(GDH) + Area(HAE)]
= Area(ABCD) – $\frac{1}{4}$[Area(ABC)+ Area(BCD) + Area(CDA) + Area(DAB)]
= Area(ABCD) – $\frac{1}{2}$Area(ABCD) = $\frac{1}{2}$Area(ABCD).

b) Let E = $(0,0)$, F = $(0,-1)$, G = $(1,-1)$, H = $(1,0)$. Find all points A = (p,q) in the first quadrant such that E, F, G and H respectively are the midpoints of the sides AB, BC, CD and DA of a convex quadrilateral ABCD.

If A = (p,q) is such a point, then E = (0,0) being the midpoint of AB is equivalent to having B = $(-p,-q)$. Similarly we get C = $(p,q-2)$, D = $(2-p,-q)$. In particular AC = BD = 2, AC is vertical and BD horizontal. By the reasoning in part a), these facts imply that the quadrilateral constructed from the midpoints of the sides of ABCD is a square of side 1. So we just need to ensure that the listed coordinates make ABCD into a convex quadrilateral. This happens if and only if p,q are both positive (which is given) and < 1. It is easy to see that these conditions are sufficient to make ABCD a convex quadrilateral. For necessity see the following (pictures will help). If $p > 1$ then A will be to the right of H and so D to the left of H. If $q > 1$, then B will be below F and so C will be above F. If p or $q = 1$, then three of the points A, B, C, D become collinear. In all cases ABCD will not be a convex quadrilateral. If both $p,q > 1$, ABCD will even be self-intersecting.

B3. a) We want to choose subsets $A_1, A_2, \ldots, A_k$ of $\{1,2,\ldots,n\}$ such that any two of the chosen subsets have nonempty intersection. Show that the size k of any such collection of subsets is at most 2^{n-1}.

If a set A is in such a collection $\mathcal{C}$, then the complement of A cannot be in $\mathcal{C}$. Therefore $|\mathcal{C}| \leq \frac{1}{2}$(total number of subsets of $\{1,2,\ldots,n\}$) $= \frac{1}{2}2^n = 2^{n-1}$.

b) For $n > 2$ show that we can always find a collection of 2^{n-1} subsets $A_1, A_2, \ldots$ of $\{1,2,\ldots,n\}$ such that any two of the A_i intersect, but the intersection of all A_i is empty.

There are many ways to build such a collection, e.g., take all 2^{n-1} subsets of $\{1,2,\ldots,n\}$ containing 1, remove the singleton set $\{1\}$ and instead include its complement. –OR– Note that for $n = 3$, the four sets $\{1,2\}, \{2,3\}, \{1,3\}, \{1,2,3\}$ give a (unique) solution. For $n > 3$ take the union of each of these 4 sets with all 2^{n-3} subsets of $\{4,\ldots,n\}$. –OR– For $n = 2k+1$, take all subsets of $\{1,2,\ldots,n\}$ of size $> k$. Any two of these will intersect. Now use $\binom{n}{i} = \binom{n}{n-i}$. For $n = 2k$, take all subsets of size $> k$ along with half the subsets of size k, namely those containing a fixed number. (Check the details.)

B4. Define

$$x = \sum_{i=1}^{10} \frac{1}{10\sqrt{3}} \frac{1}{1+(\frac{i}{10\sqrt{3}})^2} \qquad \text{and} \qquad y = \sum_{i=0}^{9} \frac{1}{10\sqrt{3}} \frac{1}{1+(\frac{i}{10\sqrt{3}})^2}.$$

Show that a) $x < \frac{\pi}{6} < y$ and b) $\frac{x+y}{2} < \frac{\pi}{6}$. (Hint: Relate these sums to an integral.)

a) Let $f(t) = 1/(1+t^2)$. Then y and x are respectively the left and right hand Riemann sums for f over the interval $[0, \frac{1}{\sqrt{3}}]$ using 10 equal parts, each of width $1/10\sqrt{3}$. Since f is a positive decreasing function, y overestimates the area under f over the given interval and x underestimates it. The area under f over $[0, \frac{1}{\sqrt{3}}]$ is $\int_0^{1/\sqrt{3}} f(t)dt = \tan^{-1}(t)|_0^{1/\sqrt{3}} = \frac{\pi}{6}$, so $x < \frac{\pi}{6} < y$. *Note.* Different normalizations are possible for f, e.g., the more simpleminded choice $f(t) = \frac{1}{10\sqrt{3}} \frac{1}{1+(\frac{t}{10\sqrt{3}})^2}$ considered over the interval [0,10] will work too.

b) $\frac{x+y}{2}$ can be interpreted as the sum of areas of 10 trapezoids as follows. Dividing $[0, \frac{1}{\sqrt{3}}]$ into 10 equal parts, let the i-th subinterval be $[t_{i-1}, t_i]$ with $i = 0, 1, \ldots, 10$. Then the i-th trapezoid has base $[t_{i-1}, t_i]$ and it has two vertical sides, the left one of height $f(t_{i-1})$ and the right one of height $f(t_i)$ (draw a picture and see). So we have to prove that the total area of trapezoids is less than the area under f. For this we should check concavity of f (draw pictures and see why). Check that over the interval $(0, \frac{1}{\sqrt{3}})$, we have $f''(t) = \frac{6t^2-2}{(1+t^2)^3} < 0$, so f is concave down and hence each trapezoid lies completely below the graph of f.

B5. Using the steps below, find the value of $x^{2012} + x^{-2012}$, where $x + x^{-1} = \frac{\sqrt{5}+1}{2}$.

a) For any *real* r, show that $|r + r^{-1}| \geq 2$. What does this tell you about the given x?

Because of the absolute value we may assume that $r > 0$ by replacing r with $-r$ if necessary. Now use AM-GM inequality or the fact that $(\sqrt{r} - \sqrt{1/r})^2 \geq 0$. Since $x + x^{-1} = \frac{\sqrt{5}+1}{2} < 2$, given x must be a non-real (complex) number.

b) Show that $\cos(\frac{\pi}{5}) = \frac{\sqrt{5}+1}{4}$, e.g. compare $\sin(\frac{2\pi}{5})$ and $\sin(\frac{3\pi}{5})$.

Let $\theta = \frac{\pi}{5}$. Then $\sin(2\theta) = \sin(\pi - 2\theta) = \sin(3\theta)$. Using the formulas for $\sin(2\theta)$ and $\sin(3\theta)$, canceling $\sin\theta$ (it is nonzero) and substituting $\sin^2\theta = 1 - \cos^2\theta$, gives the quadratic equation $4\cos^2\theta - 2\cos\theta - 1 = 0$. Since $\cos\theta > 0$, we get $\cos\theta = \frac{\sqrt{5}+1}{4}$.

c) Combine conclusions of parts a and b to express x and therefore the desired quantity in a suitable form.

Let $x = de^{i\alpha} = d(\cos\alpha + i\sin\alpha)$. Then $x^{-1} = d^{-1}e^{-i\alpha} = d^{-1}(\cos\alpha - i\sin\alpha)$. Adding and using that $x + x^{-1} = \frac{\sqrt{5}+1}{2} = 2\cos(\frac{\pi}{5})$, we get $d = 1$ and $\alpha = \pm\theta$. So $x = e^{\pm\frac{i\pi}{5}}$ and $x^{2012} + x^{-2012} = 2\cos(\frac{2012\pi}{5}) = 2\cos(402\pi + \frac{2\pi}{5}) = 2\cos(\frac{2\pi}{5}) = 2\cos^2(\frac{\pi}{5}) - 1 = \frac{\sqrt{5}-1}{2}$.

B6. For $n > 1$, a *configuration* consists of $2n$ distinct points in a plane, n of them red, the remaining n blue, with no three points collinear. A *pairing* consists of n line segments, each with one blue and one red endpoint, such that each of the given $2n$ points is an endpoint of exactly one segment. Prove the following.

a) For any configuration, there is a pairing in which no two of the n segments intersect. (Hint: consider total length of segments.)

For any configuration, there are only finitely many pairings. Choose one with least possible total length of segments. Here no two of the n segments can interest, because if RB and $R'B'$ intersect in point X then we get a contradiction as follows. Using triangle inequality in triangles RXB' and $R'XB$, we get $RB' + R'B < RB + R'B'$ (draw a picture). So replacing RB and $R'B'$ with $R'B$ and RB' would give a pairing with smaller total length.

b) Given n red points (no three collinear), we can place n blue points such that any pairing in the resulting configuration will have two segments that do not intersect. (Hint: First consider the case $n = 2$.)

For $n = 2$, place the two blue points on opposite sides of the line passing through the given two red points. There are two possible pairings and the two segments in either one do not intersect. We use a similar idea in general. Given n red points, find a triangle ABC such that A is a red point and all other red points are inside triangle ABC. (This is always possible. Why?) Place one blue point at B and all other blue points in the region opposite to triangle ABC at vertex C. (More precisely, let C be between A and A' and also between B and B'. Place the remaining blue points inside triangle $A'CB'$.) Now in any pairing, if A and B are connected, then AB will not intersect any other segment. Otherwise the two segments having A and B as vertices will not intersect. Draw a picture to see this.

B7. A sequence of integers c_n starts with $c_0 = 0$ and satisfies $c_{n+2} = ac_{n+1} + bc_n$ for $n \geq 0$, where a and b are integers. For any positive integer k with $gcd(k, b) = 1$, show that c_n is divisible by k for infinitely many n.

Consider pairs of consecutive entries of the sequence modulo k, i.e., $(\bar{c}_n, \bar{c}_{n+1})$, where $\bar{a}$ denotes a modulo k. Since there are only finitely many possibilities (namely k^2), some pair of consecutive residues will repeat. Suppose $(\bar{c}_i, \bar{c}_{i+1}) = (\bar{c}_{i+p}, \bar{c}_{i+p+1})$ for some i. We will show that in fact the previous equation holds for *all* i, i.e., whole sequence of consecutive pairs is periodic. This will prove in particular that $(\bar{c}_0, \bar{c}_1) = (\bar{c}_p, \bar{c}_{p+1}) = (\bar{c}_{2p}, \bar{c}_{2p+1}) = \cdots$. Since $c_0 = 0$ is divisible by k, so is c_{ip} for all i.

The equation $c_{n+2} = ac_{n+1} + bc_n$ shows that $\bar{b}\bar{c}_n = \bar{c}_{n+2} - \bar{a}\bar{c}_{n+1}$. Now $gcd(k, b) = 1$ means b is invertible modulo k, i.e., there is a b' with $\bar{b}\bar{b}' = \bar{1}$. Therefore $\bar{c}_n = \bar{b}'(\bar{c}_{n+2} - \bar{a}\bar{c}_{n+1})$. Thus knowing a pair of consecutive residues uniquely determines the previous residue (this is why we considered *pairs* of residues). Therefore $(\bar{c}_i, \bar{c}_{i+1}) = (\bar{c}_{i+p}, \bar{c}_{i+p+1})$ implies $(\bar{c}_{i-1}, \bar{c}_i) = (\bar{c}_{i+p-1}, \bar{c}_{i+p})$ and (by the given recurrence) $(\bar{c}_{i+1}, \bar{c}_{i+2}) = (\bar{c}_{i+p+1}, \bar{c}_{i+p+2})$. Thus the whole sequence $(\bar{c}_n, \bar{c}_{n+1})$ becomes periodic as soon as a single such pair repeats.

B8. Let $f(x)$ be a polynomial with integer coefficients such that for each nonnegative integer n, $f(n) =$ a perfect power of a prime number, i.e., of the form p^k, where p is prime and k a positive integer. (p and k can vary with n.) Show that f must be a constant polynomial using the following steps or otherwise.

a) If such a polynomial $f(x)$ exists, then there is a polynomial $g(x)$ with integer coefficients such that for each nonnegative integer n, $g(n) =$ a perfect power of a *fixed* prime number.

Write $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$. Then $a_0 = f(0) = p^k$ for some prime p and integer $k > 0$. Define $g(x) = f(px)$. Then $g(x)$ is a polynomial such that for each nonnegative integer n, $g(n) = f(pn) =$ a perfect power of a prime number. This prime number has to be p, because by evaluating we see that $g(n) = f(pn)$ is divisible by p.

b) Show that a polynomial $g(x)$ as in part a must be constant.

Let $g(x) = b_n x^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_0$. Then $b_0 = g(0) = p^k$. Consider $g(mp^{k+1}) = b_n(mp^{k+1})^n + b_{n-1}(mp^{k+1})^{n-1} + \cdots + b_1(mp^{k+1}) + p^k$. Clearly for each non-negative integer m, this expression is divisible by p^k, but not by p^{k+1} (since it is p^k modulo p^{k+1}). This forces $g(mp^{k+1}) = p^k$ for all m, since it must be a perfect power of p. Thus the polynomial g takes the value p^k infinitely often, so it must be identically equal to p^k. (Otherwise the polynomial $g(x) - p^k$ would have infinitely many roots.) To finish the problem, note that since $g(x) = f(px)$ is constant, $f(x)$ must be constant by the same logic.

B9. Let N be the set of *non-negative* integers. Suppose $f : N \to N$ is a function such that $f(f(f(n))) < f(n+1)$ for every $n \in N$. Prove that $f(n) = n$ for all n using the following steps or otherwise.

a) If $f(n) = 0$, then $n = 0$.

Let $f(n) = 0$. If $n > 0$, then $n - 1$ is in the domain of f and $f(f(f(n-1))) < f(n) = 0$, which is a contradiction, since 0 is the smallest possible value of f. (Note that this does NOT prove that $f(0) = 0$, only that if $f(\text{some } n) = 0$, then that $n = 0$. In fact proving $f(0) = 0$ along with part a would essentially solve the problem, see below.)

b) If $f(x) < n$, then $x < n$. (Start by considering $n = 1$.)

Induction on n. If $n = 1$, then this is just part a. Assuming the statement up to n we need to prove that if $f(x) < n + 1$, then $x < n + 1$. If $f(x) < n$, then by induction $x < n$, so $x < n + 1$. So let $f(x) = n$. If $x = 0$, we are done. Otherwise $f(f(f(x-1))) < f(x) = n$ and by using induction thrice we get in succession $f(f(x-1)) < n$, then $f(x-1) < n$ and then $x - 1 < n$, i.e., $x < n + 1$ as desired.

c) $f(n) < f(n+1)$ and $n < f(n+1)$ for all n.

Apply part b to $f(f(f(m))) < f(m+1)$ (with $x = f(f(m))$ and $n = f(m+1)$) to get

$f(f(m)) < f(m+1)$. Apply part b to this with $x = f(m)$ and $n = f(m+1)$ to get $f(m) < f(m+1)$. Again apply part b to get $m < f(m+1)$.

d) $f(n) = n$ for all n.

By part c, f is increasing and $f(n) \geq n$. If $f(n) > n$, then $f(f(n)) > f(n)$ (since f is increasing) and so $f(f(n)) > n$, i.e., $f(f(n)) \geq n+1$. Again, since f is increasing, $f(f(f(n))) \geq f(n+1)$, a contradiction.

Alternative solution after part a. Let us prove $f(0) = 0$. We know that $f(n) = 0$ implies $n = 0$, so $n > 0$ implies $f(n) > 0$. Applying this to any positive $f(k)$, we get $f(f(k)) > 0$. Denoting $f(f(k)) = x$, we therefore get $f(f(f(x-1))) < f(x) = f(f(f(k)))$. This means that for k such that $f(f(f(k)))$ is the smallest number in $\{f(f(f(n)))|n \geq 0\}$, we must have $f(k) = 0$. In particular 0 is in the range of f, so by part a $f(0) = 0$.

Since $f(n) = 0$ for no other n, we may restrict the function f by deleting 0 from the domain and the range. The resulting function would satisfy $f(f(f(n))) < f(n+1)$ for every $n > 0$. Repeat the reasoning substituting 1 (the new lowest element of the domain and the range) for 0 and conclude $f(1) = 1$. Then restrict to $n > 1$ and show $f(2) = 2$ and so on.