Solutions for 2013 Entrance Examination for BSc Programmes at CMI

Part A. (10 problems $\times$ 5 points = 50 points.) **Attempt all questions in this part before going to part B. Carefully read the details of marking scheme given below. Note that wrong answers will get negative marks!**

In each problem you have to fill in 4 blanks as directed. Points will be given based only on the filled answer, so you need not explain your answer. Each correct answer gets 1 point and having all 4 answers correct will get 1 extra point for a total of 5 points per problem. But each wrong/illegible/unclear answer will get minus 1 point. Negative points from any problem will be counted in your total score, so it is better not to guess! If you are unsure about a part, you may leave it blank without any penalty. If you write something and then want it not to count, cross it out and clearly write "no attempt" next to the relevant part.

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1. For sets A and B, let $f : A \to B$ and $g : B \to A$ be functions such that $f(g(x)) = x$ for each x. For each statement below, write whether it is TRUE or FALSE.

a) The function f must be one-to-one.
b) The function f must be onto.
c) The function g must be one-to-one.
d) The function g must be onto.

Answer: FTTF.

If $g(x_1) = g(x_2)$, then $x_1 = f(g(x_1)) = f(g(x_2)) = x_2$, so g is one-to-one. Also f is onto because each $x \in B$ is in the image of f, namely $x = f(g(x))$. The other two statements are false, e.g. by constructing an example in which A is a larger finite set than B.

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2. Let $f : \mathbb{R} \to \mathbb{R}$ be a function, where $\mathbb{R}$ is the set of real numbers. For each statement below, write whether it is TRUE or FALSE.

a) If $|f(x) - f(y)| \leq 39|x - y|$ for all x, y then f must be continuous everywhere.
b) If $|f(x) - f(y)| \leq 39|x - y|$ for all x, y then f must be differentiable everywhere.
c) If $|f(x) - f(y)| \leq 39|x - y|^2$ for all x, y then f must be differentiable everywhere.
d) If $|f(x) - f(y)| \leq 39|x - y|^2$ for all x, y then f must be constant.

Answer: TFTT

In parts a and b, we have $|f(x) - f(a)|$ sandwiched between $\pm 39|x - a|$. As $x \to a$, $\pm 39|x - a| \to 0$ and hence $f(x) - f(a) \to 0$, so f is continuous. But it need not be differentiable, e.g. $f(x) = |x|$ satisfies $f(x) - f(y) = |x| - |y| \leq |x - y| \leq 39|x - y|$. But f is not differentiable at 0.

In parts c and d, we have $|\frac{f(x)-f(a)}{x-a}| \leq 39|x - a|$, so by reasoning as for part a, we have $\lim_{x\to a} \frac{f(x)-f(a)}{x-a} = 0$, i.e., $f'(a) = 0$ for all a, so f is a constant function.

3. Let S be a circle with center O. Suppose A, B are points on the circumference of S with $\angle AOB = 120°$. For triangle AOB, let C be its circumcenter and D its orthocenter (i.e., the point of intersection of the three lines containing the altitudes). For each statement below, write whether it is TRUE or FALSE.

a) The triangle AOC is equilateral.
b) The triangle ABD is equilateral.
c) The point C lies on the circle S.
d) The point D lies on the circle S.

Answer: TTTT

Draw a picture and see that the bisector of $\angle AOB$ splits this angle into two angles of $60°$ each and meets the circle, say in point C'. Now the triangles OAC' and OBC' are both equilateral, so $AC' = OC' = BC'$, making $C' = C$, the cirumcenter of triangle AOB. Similarly, letting CD' be a diameter of the circle S, it is easy to deduce that $\angle AOD' = \angle BOD' = 120°$ and that triangle ABD' is also equilateral with O as its centroid. Hence $CD' \perp AB$, line $BO \perp AD'$ and line $AO \perp BD'$, making $D' = D$, the orthocenter of triangle AOB.

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4. A polynomial $f(x)$ with real coefficients is said to be a sum of squares if we can write $f(x) = p_1(x)^2 + \cdots + p_k(x)^2$, where $p_1(x), \ldots, p_k(x)$ are polynomials with real coefficients. For each statement below, write whether it is TRUE or FALSE.

a) If a polynomial $f(x)$ is a sum of squares, then the coefficient of every odd power of x in $f(x)$ must be 0.
b) If $f(x) = x^2 + px + q$ has a non-real root, then $f(x)$ is a sum of squares.
c) If $f(x) = x^3 + px^2 + qx + r$ has a non-real root, then $f(x)$ is a sum of squares.
d) If a polynomial $f(x) > 0$ for all real values of x, then $f(x)$ is a sum of squares.

Answer: FTFT

For part b, complete the square to get $f(x) = x^2 + px + q = (x + \frac{p}{2})^2 + (\frac{4q-p^2}{4})$, which is a sum of squares since $4q - p^2 > 0$ due to the roots being non-real. Since p need not be 0, this disproves part a. For part d, since all roots of f are non-real and occur in conjugate pairs, $f(x) =$ a product of quadratic polynomials each of which is a sum of squares by part b. For part c, note that $f(x) \to -\infty$ as $x \to -\infty$, so in particular $f(x)$ takes negative values and hence can never be a sum of squares. (This applies to any odd degree polynomial.)

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5. There are 8 boys and 7 girls in a group. For each of the tasks specified below, write an expression for the number of ways of doing it. Do NOT try to simplify your answers.

a) Sitting in a row so that all boys sit contiguously and all girls sit contiguously, i.e., no girl sits between any two boys and no boy sits between any two girls.

Answer: $2 \times 8! \times 7!$ (The factor of 2 arises because the two blocks of boys and girls can switch positions.)

b) Sitting in a row so that between any two boys there is a girl and between any two girls there is a boy

Answer: $8! \times 7!$ (There is no factor of 2 because there must be a boy at each end.)

c) Choosing a team of six people from the group Answer: $\binom{15}{6}$

d) Choosing a team of six people consisting of *unequal* number of boys and girls

Answer: $\binom{15}{6} - \binom{8}{3}\binom{7}{3} = \binom{8}{6} + \binom{8}{5}\binom{7}{1} + \binom{8}{4}\binom{7}{2} + \binom{8}{2}\binom{7}{4} + \binom{8}{1}\binom{7}{5} + \binom{7}{6}$

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6. Calculate the following integrals whenever possible. If a given integral does not exist, state so. Note that $[x]$ denotes the integer part of x, i.e., the unique integer n such that $n \le x < n+1$.

a) $\int_1^4 x^2 dx = \frac{x^3}{3}|_1^4 = 21$ using the fundamental theorem of calculus.

b) $\int_1^3 [x]^2 dx = 1(1^2) + 1(2^2) = 5 =$ area under the piecewise constant function $[x]^2$

c) $\int_1^2 [x^2] dx = 1(\sqrt{2}-1) + 2(\sqrt{3}-\sqrt{2}) + 3(2-\sqrt{3}) = 5 - \sqrt{2} - \sqrt{3}$ since the function $[x]^2$ is constant on intervals $[1,\sqrt{2}), [\sqrt{2},\sqrt{3}), [\sqrt{3},2)$, taking values $1, 2, 3$ respectively.

d) $\int_{-1}^1 \frac{1}{x^2} dx = 2\lim_{t\to 0^+} \int_t^1 \frac{1}{x^2} dx = 2\lim_{t\to 0^+}(-1+\frac{1}{t}) = \infty$. The fundamental theorem does not apply over the interval $[-1,1]$ because $\frac{1}{x^2}$ goes to ∞ in the interval. It is also ok to answer that the integral does not exist (as a real number).

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7. Let A, B, C be angles such that e^{iA}, e^{iB}, e^{iC} form an equilateral triangle in the complex plane. Find values of the given expressions.

a) $e^{iA} + e^{iB} + e^{iC} = 0$ by taking the vector sum of the three points on the unit circle.

b) $\cos A + \cos B + \cos C = 0 =$ real part of $e^{iA} + e^{iB} + e^{iC}$, which is 0 by part a.

c) $\cos 2A + \cos 2B + \cos 2C = 0$ because the points $e^{2iA}, e^{2iB}, e^{2iC}$ on the unit circle also form an equilateral triangle in the complex plane, since taking $B = A + (2\pi/3), C = A + (4\pi/3)$, we get $2B = 2A + (4\pi/3)$ and $2C = 2A + (8\pi/3) = 2A + (2\pi/3) + 2\pi$ and the last term 2π does not change the position of the point.

d) $\cos^2 A + \cos^2 B + \cos^2 C = \frac{3}{2}$ because, using the formula for $\cos 2\theta$ in part c, we get $\cos^2 A + \cos^2 B + \cos^2 C = \sin^2 A + \sin^2 B + \sin^2 C$ and the sum of the LHS and the RHS in this equation is 3.

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8. Consider the quadratic equation $x^2 + bx + c = 0$, where b and c are chosen randomly from the interval [0,1] with the probability uniformly distributed over all pairs (b, c). Let $p(b) =$ the probability that the given equation has a real solution for *given* (fixed) value of b. Answer the following questions by filling in the blanks.

a) The equation $x^2 + bx + c = 0$ has a real solution if and only if $b^2 - 4c$ is ≥ 0.

b) The value of $p(\frac{1}{2})$, i.e., the probability that $x^2 + \frac{x}{2} + c = 0$ has a real solution is

Answer: $\frac{1}{16}$ since a real solution occurs precisely when $b^2 - 4c = \frac{1}{4} - 4c \geq 0$, i.e., $0 \leq c \leq \frac{1}{16}$, which is $\frac{1}{16}^{\text{th}}$ fraction of the interval $[0, 1]$ over which c ranges.

c) As a function of b, is $p(b)$ increasing, decreasing or constant?

Answer: increasing, because $b^2 - 4c \geq 0$ if and only if $0 \leq c \leq \frac{b^2}{4}$, so $p(b) = \frac{b^2}{4}$, which is increasing for $0 \leq b \leq 1$.

d) As b and c both vary, what is the probability that $x^2 + bx + c = 0$ has a real solution?

Answer: This is the fraction of the area of the unit square $[0, 1] \times [0, 1]$ that is occupied by the region $b^2 - 4c \geq 0$, i.e., it is the area *under* the parabola $c = \frac{b^2}{4}$ from $b = 0$ to $b = 1$, which is $\int_0^1 \frac{b^2}{4} db = \frac{1}{12}$.

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9. Let $\mathbb{R}$ = the set of real numbers. A continuous function $f : \mathbb{R} \to \mathbb{R}$ satisfies $f(1) = 1$, $f(2) = 4$, $f(3) = 9$ and $f(4) = 16$. Answer the independent questions below by choosing the correct option from the given ones.

a) Which of the following values must be in the range of f ?
Options: 5 25 both neither

Answer: 5, by the intermediate value theorem, e.g., over the interval [2,3]. Also $f(x)$ need not take the value 25, e.g., take $f(x) = x^2$ for $x < 4$ and $f(x) = 16$ for $x \geq 4$.

b) Suppose f is differentiable. Then which of the follwing intervals must contain an x such that $f'(x) = 2x$? **Options:** (1,2) (2,4) both neither

Answer: both

c) Suppose f is twice differentiable. Which of the following intervals must contain an x such that $f''(x) = 2$? **Options:** (1,2) (2,4) both neither

Answer: (2,4)

d) Suppose f is a polynomial, then which of the following are possible values of its degree?
Options: 3 4 both neither

Answer: 4

For parts b,c and d, let $g(x) = f(x) - x^2$. We have $g(1) = g(2) = g(3) = g(4) = 0$. For part b, applying Rolle's theorem to $g(x)$ gives $g'(x) = 0$ for some values of x in each of the intervals $(1, 2), (2, 3), (3, 4)$. For these values of x, we have $f'(x) = g'(x) + 2x = 2x$.

Far part c, take from part b values $r \in (2, 3)$ and $s \in (3, 4)$ with $g'(r) = 0 = g'(s)$. Applying Rolle's theorem to $g'(x)$ in the interval $[r, s]$, we get for some $x \in (r, s) \subset (2, 4)$ the equality $g''(x) = 0$ and so $f''(x) = g''(x) + 2 = 2$. There need not be an $x \in (1, 2)$ with $f''(x) = 2$, i.e., $g''(x) = 0$. There are many ways to arrange this, for example let $g(x) = \sin(\pi x)$. Then

$g''(x) = -\pi^2 \sin(\pi x)$, which is 0 only when x is an integer, in particular $g''(x) \neq 0$ for any $x \in (1, 2)$.

For part d, note that $g(x)$, now being a polynomial vanishing at 1, 2, 3 and 4, must be divisible by $(x-1)(x-2)(x-3)(x-4)$. So $g(x)$, if non-zero, must have degree at least 4. Thus $f(x) = x^2$ or a polynomial of degree at least 4.

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10. Let

$$f(x) = \frac{x^4}{(x-1)(x-2)\cdots(x-n)}$$

where the denominator is a product of n factors, n being a positive integer. It is also given that the X-axis is a horizontal asymptote for the graph of f. Answer the independent questions below by choosing the correct option from the given ones.

a) How many vertical asymptotes does the graph of f have?
Options: n less than n more than n impossible to decide

Answer: n, at $x = 1, 2, \ldots, n$.

b) What can you deduce about the value of n ?
Options: $n < 4$ $n = 4$ $n > 4$ impossible to decide

Answer: $n > 4$, because $\lim_{x \to \pm\infty} f(x) = 0$ and for this to happen, the degree of the denominator of $f(x)$ must be greater than that of the numerator.

c) As one travels along the graph of f from left to right, at which of the following points is the sign of $f(x)$ guaranteed to change from positive to negative?
Options: $x = 0$ $x = 1$ $x = n-1$ $x = n$

Answer: $x = n-1$, because $f(x)$ is positive for $x > n$ and $f(x)$ changes sign precisely when it passes through $x = 1, 2 \ldots, n$. Note that the sign of $f(x)$ for $x < 0$ and for $x \in (0, 1)$ depends on the parity of n.

d) How many inflection points does the graph of f have in the region $x < 0$?
Options: none 1 more than 1 impossible to decide
(Hint: Sketching is better than calculating.)

Answer: more than 1. Note that $f(x) = 0$ only at $x = 0$, with multiplicity 4. Without loss of generality, let n be even. (If n is odd, the reasoning is completely parallel, see note at the end.) Now $f(x) > 0$ for $x < 1$ except at $x = 0$ and f has all derivatives for $x < 1$. Due to the multiple root at $x = 0$, the graph of f must be concave up (i.e. $f''(x) > 0$) near $x = 0$. Further, as $x \to -\infty$, the values of $f(x)$ stay positive and $\to 0$. Therefore, as one traces the graph leftward from the origin, it must become concave down at least once and eventually concave up again so as to approach the X-axis from above. (Note: If n is odd, $f(x) < 0$ for $x < 1$ except at $x = 0$. As one traces the graph leftward from the origin, the function is initially as well as eventually concave *down* and must be concave up at least once in-between so as to approach the X-axis from below.)

Part B. (Problems 1–4 $\times$ 15 points + problems 5–6 $\times$ 20 points = 100 points.) Solve these problems in the space provided for each problem after this page. You may solve only part of a problem and get partial credit. **Clearly explain your entire reasoning. No credit will be given without reasoning.**

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1. In triangle ABC, the bisector of angle A meets side BC in point D and the bisector of angle B meets side AC in point E. Given that DE is parallel to AB, show that AE = BD and that the triangle ABC is isosceles.

Answer: $\angle EAD = \angle DAB = \angle EDA$, the first equality because AD bisects $\angle EAB$ and the second because alternate angles made by line AD intersecting parallel lines DE and AB are equal. Thus $\triangle EAD$ is isosceles with $EA = ED$. Similarly $ED = DB$ using the fact that BE bisects $\angle DBA$ also intersects parallel lines DE and AB. Therefore $EA = ED = DB$. Now by the basic proportionality theorem, $\frac{CE}{EA} = \frac{CD}{DB}$. As the denominators EA and DB are equal, the numerators must be equal as well, i.e., $CE = CD$. Finally, $CA = CE + EA = CD + DB = CB$, so $\triangle ABC$ is isosceles.

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2. A curve C has the property that the slope of the tangent at any given point (x, y) on C is $\frac{x^2+y^2}{2xy}$.

a) Find the general equation for such a curve. Possible hint: let $z = \frac{y}{x}$.

b) Specify all possible shapes of the curves in this family. (For example, does the family include an ellipse?)

Answer: The defining property of the curve C is equivalent to the differential equation $\frac{dy}{dx} = \frac{x^2+y^2}{2xy} = \frac{1}{2}(\frac{x}{y} + \frac{y}{x})$. It is convenient to let $z = y/x$, so the equation becomes $\frac{dy}{dx} = \frac{1}{2}(\frac{1}{z} + z)$. To get this in terms of only x and z, differentiate $z = y/x$ with respect to x to get $\frac{dz}{dx} = \frac{1}{x}\frac{dy}{dx} - \frac{y}{x^2} = \frac{1}{x}(\frac{dy}{dx} - z) = \frac{1}{x}(\frac{1}{2}(\frac{1}{z} + z) - z) = \frac{1}{x}\frac{1-z^2}{2z}$, where we have substituted for $\frac{dy}{dx}$ using the differential equation and then simplified. Separating the variables and integrating, we get $\int \frac{dx}{x} = \int \frac{2zdz}{1-z^2}$, which gives $\log|x| = -\log|1 - z^2| +$ a constant, i.e., $\log|1 - z^2| = -\log|x| + K = \log|x|^{-1} + K$. Exponentiating, we get $1 - z^2 = \pm\frac{e^K}{x} = \frac{c}{x}$, where c is a *nonzero* constant. Substituting $z = y/x$, we get $1 - \frac{y^2}{x^2} = \frac{c}{x}$, i.e., $x^2 - y^2 = cx$. To be precise, we have to delete the points $(0, 0)$ and $(c, 0)$ from this solution, because for the given equation $\frac{dy}{dx} = \frac{x^2+y^2}{2xy}$ to make sense, both x and y must be nonzero. If the equation were given as $2xy\frac{dy}{dx} = x^2 + y^2$, then this issue would not arise.

To see the shape of the curve, complete the square to get $(x - \frac{c}{2})^2 - y^2 = \frac{c^2}{4}$, which is a hyperbola when $c \neq 0$. (*Note*: By differentiating $x^2 - y^2 = cx$, it is easy to see that $\frac{dy}{dx} = \frac{2x-c}{y} = \frac{x^2+y^2}{2xy}$ and that this holds even when $c = 0$. Thus we get the two straight lines $y = \pm x$ also as solutions. The reason the above answer missed this possibility was because we put $1 - z^2$ in the denominator while separating variables, which precludes $z = \pm 1$, i.e., $y = \pm x$. To be precise, even here we have to delete the origin from the two lines.)

3. A positive integer N has its first, third and fifth digits equal and its second, fourth and sixth digits equal. In other words, when written in the usual decimal system it has the form $xyxyxy$, where x and y are the digits. Show that N cannot be a perfect power, i.e., N cannot equal a^b, where a and b are positive integers with $b > 1$.

Answer: We have $N = (10^5 + 10^3 + 10)x + (10^4 + 10^2 + 1)y = 10101(10x + y) = 3 \times 7 \times 13 \times 37 \times (10x + y)$. Therefore for N to be a perfect power, the primes 3,7,13,37 must all occur (and in fact with equal power) as factors in the prime factorization of $10x + y$. In particular $10x + y \geq 10101$. But since x and y are digits, each is between 0 and 9, so $10x + y \leq 99$. So N cannot be a perfect power.

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4. Suppose $f(x)$ is a function from $\mathbb{R}$ to $\mathbb{R}$ such that $f(f(x)) = f(x)^{2013}$. Show that there are infinitely many such functions, of which exactly four are polynomials. (Here $\mathbb{R}$ = the set of real numbers.)

Answer: If f is a polynomial, then we make two cases. (i) If $f(x) =$ a constant c, then the given condition is equivalent to $c = c^{2013}$, which happens precisely for three values of c, namely $c = 0, 1, -1$ (since we have $c(c^{2012} - 1) = 0$, so $c = 0$ or $c^{2012} = 1$). Thus there are three constant functions with the given property. (ii) If $f(x)$ is a non-constant polynomial, then consider its range set $A = \{f(x)|x \in \mathbb{R}\}$. Now for all $a \in A$, we have by the given property $f(a) = a^{2013}$. So the polynomial $f(x) - x^{2013}$ has all elements of A as its roots. Since there are infinitely many values in A (e.g. applying the intermediate value theorem because f is continuous), the polynomial $f(x) - x^{2013}$ has infinitely many roots and thus must be the zero polynomial, i.e., $f(x) = x^{2013}$ for all real number x.

Note: One can also deduce that the degree of f must be 0 or 2013 by equating the degrees of $f(f(x))$ and $f(x)^{2013}$. Then, in the non-constant case, it is possible to argue first that the leading coefficient is 1 and then that all other coefficients must be 0.

To find infinitely many function with the given property, define $f(0) = 0, f(1) = 1$ and $f(-1) = -1$. For every other real number x, arbitrarily define $f(x)$ to be $0, 1$ or -1. It is easy to see that any such function satisfies the given property. (Other answers are possible, e.g., more systematically, observe that $f(a) = a^{2013}$ for at least one real number a (e.g., any number in the range of f) and then this forces $f(x) = x^{2013}$ for all $x \in S = \{a^{2013^i}|i = 0, 1, 2, \ldots\}$. We use this as follows. Fix a real number a. Then *define* $f(x) = x^{2013}$ for all $x \in S = \{a^{2013^i}|i = 0, 1, 2, \ldots\}$. For all $x \notin S$, simply define $f(x) =$ any element of the set S, e.g., a itself will do.)

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5. Consider the function $f(x) = ax + \frac{1}{x+1}$, where a is a *positive* constant. Let $L =$ the largest value of $f(x)$ and $S =$ the smallest value of $f(x)$ for $x \in [0, 1]$. Show that $L - S > \frac{1}{12}$ for any $a > 0$.

Answer: Let $f(x) = ax + \frac{1}{x+1}$. We wish to understand the minimum and maximum of this function in the interval $[0, 1]$. Now $f(0) = 1$, $f(1) = a + \frac{1}{2}$ and $f'(x) = a - \frac{1}{(x+1)^2}$. Over the interval $[0, 1]$, the value of $f'(x)$ increases from $a - 1$ at $x = 0$ to $a - \frac{1}{4}$ at $x = 1$.

We should consider what happens to the sign of $f'(x)$. For this we consider the following cases.

(1) Suppose $a \leq 1/4$. Because $1/(x+1)^2 \geq 1/4$ on the interval $[0,1]$, $f'(x) \leq 0$, so the maximum is at 0 and the minimum is at $x = 1$. So the difference is $1 - (1/2 + a) = 1/2 - a \geq 1/4 \geq 1/12$.

(2) Suppose $a \geq 1$. Then $f'(x) \geq 0$ on the interval $[0,1]$, so maximum is at 1 and minimum at 0. We get $a + 1/2 - 1 = a - 1/2 \geq 1/2 \geq 1/12$.

(3) Suppose $1/4 \leq a \leq 1$. Now $f'(x) = 0$ at $\tilde{x} = \frac{1}{\sqrt{a}} - 1$. For this range of a, $\tilde{x} \in [0,1]$. In the interval $[0, \tilde{x}]$, $f'(x) \leq 0$ and in the interval $[\tilde{x}, 1]$, $f'(x) \geq 0$. Now we make two sub-cases depending on at which endpoint the maximum occurs.

(3i) Suppose $1/4 \leq a \leq 1/2$. Then $f(0) \geq f(1)$. So minimum is at $\tilde{x}$, maximum is at $x = 0$. $f(\tilde{x}) = \sqrt{a} - a + \sqrt{a} = 2\sqrt{a} - a$. So the difference between maximum and minimum is $1 + a - 2\sqrt{a} = (1 - \sqrt{a})^2$. This is smallest when $\sqrt{a}$ is closest to 1 and so $(1 - \sqrt{a})^2 \geq (1 - 1/\sqrt{2})^2 = 3/2 - \sqrt{2}$. This is bigger than $1/12$ since $(\frac{3}{2} - \frac{1}{12}) = 17/12$ and $17^2 = 289 \geq 2 \times 12^2$.

(3ii) Suppose $1/2 \leq a \leq 1$. Now $f(1) \geq f(0)$. Max is at 1 and minimum is at $\tilde{x}$. The difference is $a + 1/2 - \sqrt{a} + a - \sqrt{a} = 2a - 2\sqrt{a} + 1/2 = (\sqrt{2a} - \frac{1}{\sqrt{2}})^2$. By a calculation similar to the above it is bigger than $1/12$.

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6. Define $f_k(n)$ to be the sum of all possible products of k distinct integers chosen from the set $\{1, 2, \ldots, n\}$, i.e.,

$$f_k(n) = \sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq n} i_1 i_2 \ldots i_k \, .$$

a) For $k > 1$, write a recursive formula for the function f_k, i.e., a formula for $f_k(n)$ in terms of $f_\ell(m)$, where $\ell < k$ or ($\ell = k$ and $m < n$).

b) Show that $f_k(n)$, as a function of n, is a polynomial of degree $2k$.

c) Express $f_2(n)$ as a polynomial in variable n.

Answer: a) Break up the terms in the definition of $f_k(n)$ into two groups: the terms in which $i_k = n$ add up to $n f_{k-1}(n-1)$ and the remaining terms, i.e., the ones in which $i_k \leq n - 1$, add up to $f_k(n-1)$. So we get $f_k(n) = n f_{k-1}(n-1) + f_k(n-1)$.

c) By part a we have $f_2(n) - f_2(n-1) = n f_1(n-1) = n \times \frac{n(n-1)}{2} = \frac{1}{2}(n^3 - n^2)$. Similarly $f_2(n-1) - f_2(n-2) = \frac{1}{2}((n-1)^3 - (n-1)^2)$ and so on up to $f_2(2) - f_2(1) = \frac{1}{2}(2^3 - 2^2)$. Note that $f_2(1) = 0$, which we may also write as $\frac{1}{2}(1^3 - 1^2)$. Adding up, we get for any $n \geq 1$, $f_2(n) = \sum_{j=1}^{j=n} \frac{1}{2}(j^3 - j^2) = \frac{1}{2}(\frac{n^2(n+1)^2}{4} - \frac{n(n+1)(2n+1)}{6})$, where we have used standard formulas for the sum of first n cubes and of first n squares.

b) We prove the statement by induction on k. First $f_1(n) = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$, a polynomial of degree 2 as desired. For $k > 1$, we have by part a the equation $f_k(n) - f_k(n-1) = nf_{k-1}(n-1)$. The right hand side is a polynomial of degree $1 + 2(k-1) = 2k-1$, where $2(k-1)$ is the degree of $f_{k-1}(n-1)$ by induction and the added 1 comes from the factor n. Since successive differences in the values of f_k are given by a polynomial of degree $2k-1$, the function f_k on positive integers is given by a polynomial of degree 1 more, i.e., of degree $2k$.

Note: The previous statement is a standard fact, which can be explained as follows. (1) If we *assume* that $f_k(n)$ is a polynomial, then its degree is easily found, because for any polynomial f of degree m, its "successive difference" function $f(x)-f(x-1)$ is a polynomial of degree $m-1$. (Reason: If the leading term of $f(x)$ is ax^m, then the leading term in $f(x) - f(x-1)$ is amx^{m-1}, as seen by expanding the power of $x-1$ in $ax^m - a(x-1)^m$. The remaining terms in $f(x) - f(x-1)$ do not matter because by expanding powers of $x-1$ in them and simplifying, we only get monomials of degree $< m-1$.) (2) In fact, based on the difference equation, $f_k(n)$ must be a polynomial in the variable n. This is a consequence of the following well-known fact.

Claim: given a polynomial $h(x)$ of degree d, there is a polynomial $g(x)$ of degree $d+1$ such that $g(x) - g(x-1) = h(x)$. *Proof*: Induction on d, the degree of h. If $h(x) = c$, a constant, then $g(x) = cx$ works. Now for $d > 1$, it is enough to find a polynomial $g(x)$ such that $g(x) - g(x-1) = x^d$ (because if $h(x) = cx^d + \tilde{h}(x)$, where $\tilde{h}$ has degree $< d$, by induction we find $\tilde{g}$ for $\tilde{h}$ and then $cg(x) + \tilde{g}(x)$ works for $h(x)$). To find such $g(x)$, notice that for $g_1(x) = x^{d+1}$, we have $h_1(x) = g_1(x) - g_1(x-1) = (d+1)x^d + h_2(x)$, where $h_2(x)$ is a polynomial of degree $d-1$. By induction $h_2(x) = g_2(x) - g_2(x-1)$ for a polynomial $g_2(x)$ of degree d. Now $g(x) = \frac{1}{d+1}(g_1(x) - g_2(x))$ works.