CHENNAI MATHEMATICAL INSTITUTE

Undergraduate Programme in Mathematics and Computer Science/Physics
Common Entrance Examination
15 May 2014

Enter your *Registration Number* here: **CMIUG ID–** ☐☐☐☐☐☐ OR here: **UG–** ☐☐☐☐

Enter the name of the city where you are writing this test: ☐☐☐☐☐☐☐☐☐☐☐

IMPORTANT INSTRUCTIONS!

- **Ensure that this booklet has all 13 printed sheets containing the following**: this cover page, 12 questions in part A (pages 1-2), the answer sheet for part A (page 3), 6 questions in part B (pages 4-5) and individual answer sheets for each question in part B (pages 6-12). From page 6 to 10 one side is intentionally left blank after each numbered page. For rough work use the blank pages at the end.
- **Time allowed is 3 hours. Total points 130 = 45 points for part A + 85 points for part B.**
- **Part A will be used for screening.** Part B will be graded only if your score a certain minimum in part A. This minimum will be no more than 25 points out of 45. However your scores in both parts will be used while making the final decision. Specific instructions for each part are given below.
- **Advice**: Attempt all questions in part A before going to part B. However, also ensure that you have about 2 hours (or at least 90 minutes) left for part B.

For office use only

$\sum$	Points	Remarks
Part A		
Part B		
Total		

	Points	Remarks
B1		
B2		
B3		
B4		
B5		
B6		
Total		

2014 Entrance Examination for BSc Programmes at CMI

Read the instructions on the front of the booklet carefully!

Part A. Write your final answers on page 3.

Part A is worth a total of 45 points = 3 points each for A1 to A3 + 4 points each for A4-A12. Points will be given based only on clearly legible final answers filled in on page 3.

Questions A1 to A3: *For each option given below, decide whether the statement in it is True or False. You will get 0 points for a question unless you decide all options in that question correctly. Write your answer as a sequence of letters T and F in the designated place on page 3. E.g., the answer to the question:* True or False? (A) $2+2=4$ (B) $2+2=5$ (C) $3^2+4^2=5^2$ *would be written as* **TFT**.

A1. Let α, β and c be positive numbers less than 1, with c rational and α, β irrational.

(A) The number $\alpha+\beta$ must be irrational.
(B) The infinite sum $\sum_{i=0}^{\infty} \alpha c^i = \alpha + \alpha c + \alpha c^2 + \cdots$ must be irrational.
(C) The value of the integral $\int_0^{\pi} (\beta \cos x + c)\, dx$ must be irrational.

A2. Consider the intergal $I = \int_1^{\infty} e^{ax^2+bx+c}\, dx$, where a, b, c are constants. Some combinations of values for these constants are given below and you have to decide in each case whether the integral I converges.

(A) I converges for $a=-1 \quad b=10 \quad c=100$.
(B) I converges for $a=1 \quad b=-10 \quad c=-100$.
(C) I converges for $a=0 \quad b=-1 \quad c=100$.
(D) I converges for $a=0 \quad b=0 \quad c=-100$.

A3. Given a real number x, define $g(x) = x^2 e^x$ if $x \geq 0$ and $g(x) = xe^{-x}$ if $x < 0$.

(A) The function g is continuous everywhere.
(B) The function g is differentiable everywhere.
(C) The function g is one-to-one.
(D) The range of g is the set of all real numbers.

Questions A4 to A13: *Unless specified otherwise, each answer is either a rational number or, where appropriate, one of the phrases "infinite", "does not exist", or "not possible to decide". If the answer is an integer, write it in the usual decimal form. Write non-integer rationals as ratios of two coprime integers. For questions requiring more than one answer, write all answers on the designated line in the order in which they are asked, separated by commas. In such questions you may get partial credit.*

A4. Find the slope of a line L that satisfies both of the following properties: (i) L is tangent to the graph of $y = x^3$. (ii) L passes through the point (0,200).

A5. A regular 100-sided polygon is inscribed in a circle. Suppose three of the 100 vertices are chosen at random, all such combinations being equally likely. Find the probability that the three chosen points form vertices of a right angled triangle.

A6. What is the smallest positive integer n for which $\frac{50!}{24^n}$ is *not* an integer?

A7. Let $f(x) = (x-a)(x-b)^3(x-c)^5(x-d)^7$, where a, b, c, d are real numbers with $a < b < c < d$. Thus $f(x)$ has 16 real roots counting multiplicities and among them 4 are distinct from each other. Consider $f'(x)$, i.e. the derivative of $f(x)$. Find the following, if you can: (i) the number of real roots of $f'(x)$, counting multiplicities, (ii) the number of *distinct* real roots of $f'(x)$.

A8. Let $f(x) = 7x^{32} + 5x^{22} + 3x^{12} + x^2$. (i) Find the remainder when $f(x)$ is divided by $x^2 + 1$. (ii) Find the remainder when $xf(x)$ is divided by $x^2 + 1$. In each case your answer should be a polynomial of the form $ax + b$, where a and b are constants.

A9. Let $\theta_1, \theta_2, \ldots, \theta_{13}$ be *real* numbers and let A be the average of the complex numbers $e^{i\theta_1}, e^{i\theta_2} \ldots, e^{i\theta_{13}}$, where $i = \sqrt{-1}$. As the values of θ's vary over all 13-tuples of real numbers, find (i) the maximum value attained by $|A|$, (ii) the minimum value attained by $|A|$.

A10. In each of the following independent situations we want to construct a triangle ABC satisfying the given conditions. In each case state state how many such triangles ABC exist up to congruence.

(i) $AB = 30 \quad BC = 95 \quad AC = 55$

(ii) $\angle A = 30^\circ \quad \angle B = 95^\circ \quad \angle C = 55^\circ$

(iii) $\angle A = 30^\circ \quad \angle B = 95^\circ \quad BC = 55$

(iv) $\angle A = 30^\circ \quad AB = 95 \quad BC = 55$

A11. Let $A_n =$ the area of a regular n-sided polygon inscribed in a circle of radius 1 (i.e., vertices of this regular n-sided polygon lie on a circle of radius 1). (i) Find A_{12}. (ii) Find $\lfloor A_{2014} \rfloor$, i.e., the greatest integer $\leq A_{2014}$.

A12. The total length of all 12 sides of a rectangular box is 60. (i) Write the possible values of the volume of the box. Your answer should be an interval. Now suppose in addition that the surface area of the box is given to be 56. Find, if you can, (ii) the length of the longest diagonal of the box (iii) the volume of the box.

Answers to part A

This is the only page that will be seen for grading part A, so write the answer(s) to each question on the designated line below. Write only the final answers, do not show any intermediate work. Illegible/unclear answers will not be considered.

A1. ____________ A2. ____________ A3. ____________

A4. ____

A5. ____

A6. ____

A7. ____

A8. ____

A9. ____

A10. ____

A11. ____

A12. ____

Part B. (10 points for problem B1 + 15 points each for problems B2 to B6 = 85 points.) Solve these problems in the space provided for each problem from page 6. You may solve only part of a problem and get partial credit. If you cannot solve an earlier part of a problem, you may assume it to solve a later part. **Clearly explain your entire reasoning. No credit will be given without reasoning.**

B1. Find the area of the region in the XY plane consisting of all points in the set

$$\{(x, y) | x^2 + y^2 \leq 144 \text{ and } \sin(2x + 3y) \leq 0\}.$$

B2. Let x be a real number such that $x^{2014} - x^{2004}$ and $x^{2009} - x^{2004}$ are both integers. Show that x is an integer. (Hint: it may be useful to first prove that x is rational.)

B3. (i) How many functions are there from the set $\{1, \ldots, k\}$ to the set $\{1, \ldots, n\}$?

(ii) Let P_k denote the set of all subsets of $\{1, \ldots, k\}$. Find a formula for the number of functions f from P_k to $\{1, \ldots, n\}$ such that $f(A \cup B) =$ the larger of the two integers $f(A)$ and $f(B)$. Your answer need not be a closed formula but it should be simple enough to use for given values of n and k, e.g., to see that for $k = 3$ and $n = 4$ there are 100 such functions.

Example: When $k = 2$, the set P_2 contains 4 elements: the empty set $\phi, \{1\}, \{2\}$ and $\{1, 2\}$. The function f given by $\phi \to 2, \{1\} \to 3, \{2\} \to 4, \{1, 2\} \to 4$ satisfies the given condition. But the function g given by $\phi \to 2, \{1\} \to 3, \{2\} \to 4, \{1, 2\} \to 5$ does not, because $g(\{1\} \cup \{2\}) = g(\{1, 2\}) = 5 \neq$ the larger of $g(\{1\})$ and $g(\{2\}) = \max(3, 4) = 4$.

B4. (i) Let f be continuous on $[-1, 1]$ and differentiable at 0. For $x \neq 0$, define a function g by $g(x) = \frac{f(x) - f(0)}{x}$. Can $g(0)$ be defined so that the extended function g is continuous at 0?

(ii) For f as in part (i), show that the following limit exists.

$$\lim_{r \to 0^+} \left(\int_{-1}^{-r} \frac{f(x)}{x} \, dx + \int_{r}^{1} \frac{f(x)}{x} \, dx \right)$$

(iii) Give an example showing that without the hypothesis of f being differentiable at 0, the conclusion in (ii) need not hold.

B5. (i) Let $f(x) = a_n x^n + \cdots + a_1 x + a_0$ be a polynomial, where $a_0, \ldots, a_n$ are real numbers with $a_n \neq 0$. Define the "discrete derivative of f", denoted Δf, to be the function given by $\Delta f(x) = f(x) - f(x - 1)$. Show that Δf is also a polynomial and find its leading term.

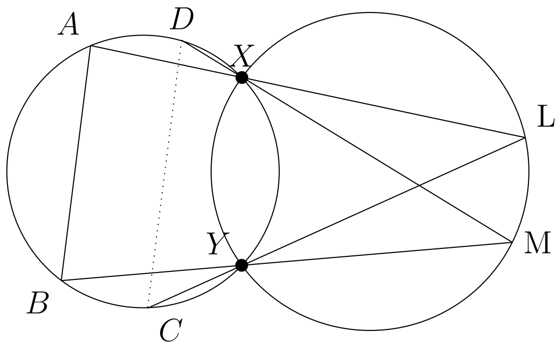
(ii) For integers $n \geq 0$, define polynomials p_n of degree n as follows: $p_0(x) = 1$ and for $n > 0$, let $p_n(x) = \frac{1}{n!} x(x - 1)(x - 2) \cdots (x - n + 1)$. So we have

$$p_0(x) = 1 \quad , \quad p_1(x) = x \quad , \quad p_2(x) = \frac{x(x - 1)}{2} \quad , \quad p_3(x) = \frac{x(x - 1)(x - 2)}{3!} \quad \cdots$$

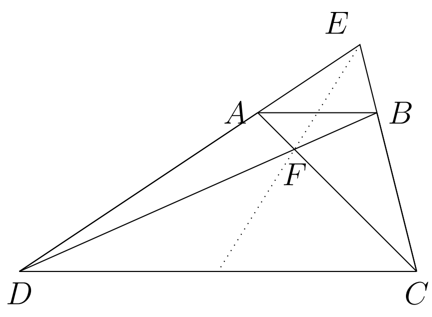
Show that for any polynomial f of degree n, there exist unique real numbers $b_0, b_1, \ldots, b_n$ such that $f(x) = \sum_{i=0}^{n} b_i p_i(x)$.

(iii) Now suppose that $f(x)$ is a polynomial such that for each integer m, $f(m)$ is also an integer. Using the above parts (or otherwise), show that for such f, the b_i obtained in part (ii) are integers.

B6. (i) See the figure below. Two circles G_1, G_2 intersect at points X, Y. Choose two other points A, B on G_1 as shown in the figure. The line segment from A to X is extended to intersect G_2 at point L. The line segment from L to Y is extended to meet G_1 at point C. Likewise the line segment from B to Y is extended to meet G_2 at point M and the segment from M to X is extended to meet G_1 at point D. Show that AB is parallel to CD.



(ii) See the figure below. A triangle CDE is given. A point A is chosen between D and E. A point B is chosen between C and E so that AB is parallel to CD. Let F denote the point of intersection of segments AC and BD. Show that the line joining E and F bisects both segments AB and segment CD. (Hint: You may use Ceva's theorem. Alternatively, you may additionally assume that the trapezium $ABCD$ is a cyclic quadrilateral and proceed.)



(iii) Using parts (i) and (ii) describe a procedure to do the following task: given two circles G_1 and G_2 intersecting at two points X and Y determine the center of each circle *using only a straightedge. Note:* Recall that a straightedge is a ruler without any markings. Given two points A, B, a straightedge allows one to construct the line segment joining A, B. Also, given any two non-parallel segments, we can use a straightedge to find the intersection point of the lines containing the two segments by extending them if necessary.

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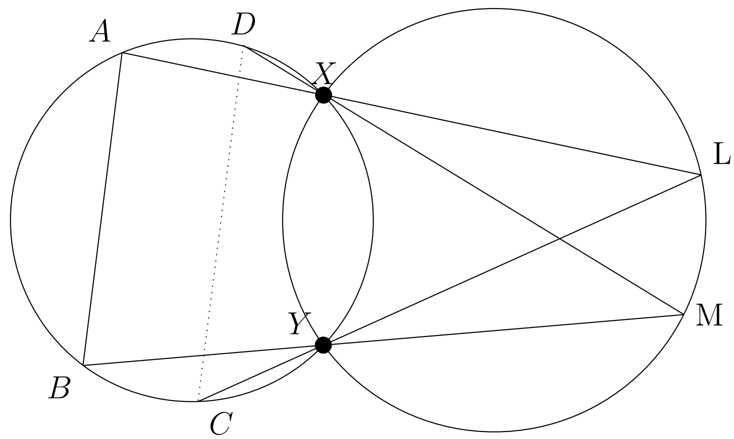

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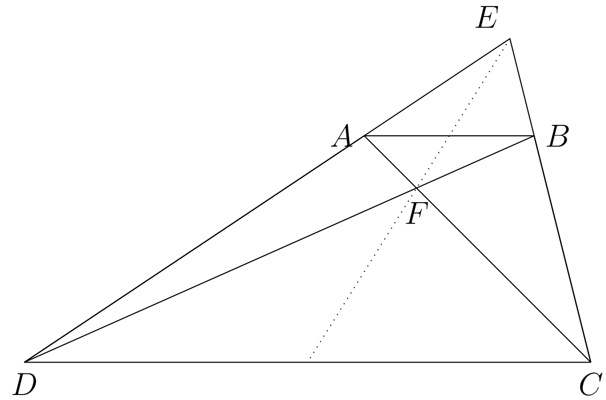

(iii) **Write your answer to this part on the back side.** Using parts (i) and (ii) describe a procedure to do the following task: given two circles G_1 and G_2 intersecting at two points X and Y determine the center of each circle *using only a straightedge*. *Note:* Recall that a straightedge is a ruler without any markings. Given two points A, B, a straightedge allows one to construct the line segment joining A, B. Also, given any two non-parallel segments, we can use a straightedge to find the intersection point of the lines containing the two segments by extending them if necessary.