Solutions to 2015 Entrance Examination for BSc Programmes at CMI

Part A Solutions

---

1. Ten people sitting around a circular table decide to donate some money for charity. You are told that the amount donated by each person was the average of the money donated by the two persons sitting adjacent to him/her. One person donated Rs. 500. Choose the correct option for each of the following two questions. Write your answers as a sequence of two letters (a/b/c/d).

   What is the total amount donated by the 10 people?

   (a) exactly Rs. 5000 (b) less than Rs. 5000 (c) more than Rs. 5000

   (d) not possible to decide among the above three options.

   What is the maximum amount donated by an individual?

   (a) exactly Rs. 500 (b) less than Rs. 500 (c) more than Rs. 500

   (d) not possible to decide among the above three options.

   Answer: exactly Rs. 5000, exactly Rs. 500. Consider the person who donated Rs. 500. Suppose the neighbor to the left donates $500 + x$. Then the one on the right donates $500 - x$. But continuing leftward, the amounts donated are $500 + 2x, 500 + 3x, \ldots$, forcing x to be 0, since you come around to the neighbor to the right.

2. Consider all finite letter-strings formed by using only two letters A and B. We consider the usual dictionary order on these strings. See below for the formal rule with examples.

   Formal rule: To compare two strings w_1 and w_2, read them from left to right. We say "w_1 is smaller than w_2" or "$w_1 < w_2$" if the first letter in which w_1 and w_2 differ is A in w_1 and B in w_2 (for example, ABAA $<$ ABB by looking at the third letters) or if w_2 is obtained by appending some letters at the end of w_1 (e.g. AB $<$ ABAA).

   For each of the statements below, state if it is true or false. Write your answers as a sequence of three letters (T for True and F for False) in correct order.

   (a) Let w be an arbitrary string. There exists a unique string y satisfying both the following properties: (i) $w < y$ and (ii) there is no string x with $w < x < y$.

   Answer: True, append A to w.

   (b) It is possible to give an infinite decreasing sequence of strings, i.e. a sequence $w_1, w_2, \ldots$, such that $w_{i+1} < w_i$ for each positive integer i.

   Answer: True. B, AB, AAB, AAAB, ...

   (c) Fewer than 50 strings are smaller than ABBABABB.

   Answer: False. There are infinitely many such strings e.g. A, AA, AAA, AAAA, ...

3. A positive integer n is called a *magic number* if it has the following property: if a and b are two positive numbers that are not coprime to n then $a + b$ is also not coprime to n. For example, 2 is a magic number, because sum of any two even numbers is also even. Which of the following are magic numbers? Write your answers as a sequence of four letters (Y for Yes and N for No) in correct order.

   (i) 129 (ii) 128 (iii) 127 (iv) 100.

   Answer: Only 128 and 127 are magic numbers. See that n is a magic number if and only if n is a power of a prime. (Otherwise, write $n = ab$ with a, b coprime.)

4. Let A, B and C be unknown constants. Consider the function $f(x)$ defined by

$$\begin{aligned} f(x) &= Ax^2 + Bx + C \quad \text{when } x \leq 0 \ , \\ &= \ln(5x+1) \quad \text{when } x > 0 \ . \end{aligned}$$

   Write the values of the constants A, B and C such that $f''(x)$, i.e., the double derivative of f, exists for all real x. If this is not possible, write "not possible". If some of the constants cannot be uniquely determined, write "not unique" for each such constant.

   Answer: The only problem is at $x = 0$. For continuity, $\ln(0+1) = C$. For $f'(0)$ to exist, f must be continuous *and* the left and right derivatives of f at $x = 0$ (which are easily seen to exist) must match, i.e. $5 = B$. For $f''(0)$ to exist, $f'(0)$ must exist and left and right derivatives of f' at $x = 0$ must match, i.e. $2A = -5^2$. So $A = -\frac{25}{2}, B = 5, C = 0$.

5. Consider the polynomial $p(x) = (x + a_1)(x + a_2) \cdots (x + a_{10})$ where a_i is a real number for each $i = 1, \ldots, 10$. Suppose all of the eleven coefficients of $p(x)$ are positive. For each of the following statements, decide if it is true or false. Write your answers as a sequence of four letters (T/F) in correct order.

   (i) The polynomial $p(x)$ must have a global minimum. (ii) Each a_i must be positive.

   (iii) All real roots of $p'(x)$ must be negative. (iv) All roots of $p'(x)$ must be real.

   Answer: All are true. (i) The degree is even, so $p(x)$ goes to $+\infty$ as $x \to \pm\infty$. So $p(x)$ must attain a global minimum somewhere by continuity. (ii) The roots of $p(x)$ are $-a_i$. By positivity of coefficients of $p(x)$, no nonnegative number is a root of $p(x)$. Thus all $-a_i$ are negative, so all $a_i > 0$. (iii + iv) All 10 roots of $p(x)$ are real and negative. There is a root of $p'(x)$ between consecutive roots of $p(x)$ (this is valid even in case of multiple roots). So all 9 roots of $p'(x)$ are real and negative as well. For negativity, one can also note that all coefficients of $p'(x)$ are positive and apply the logic in (ii) to $p'(x)$.

6. Fill in the blanks. Let C_1 be the circle with center $(-8, 0)$ and radius 6. Let C_2 be the circle with center $(8, 0)$ and radius 2. Given a point P outside both circles, let $\ell_i(P)$ be the length of a tangent segment from P to circle C_i. The locus of all points P such that $\ell_1(P) = 3\ \ell_2(P)$ is a circle with radius ___ and center at (___ , ___).

   Answer: Center = (10,0), radius = 6. Use the distance formula and the Pythagorean theorem to get $y^2+(x+8)^2-6^2 = 9(y^2+(x-8)^2-4)$. Simplifying gives $y^2+(x-10)^2 = 6^2$. Another way, assuming the locus to be a circle: note that the ratio of the radii of C_1, C_2 and that of the tangents is the same (namely 3). Now use similar triangles to see that the desired circle intersects the X-axis at coordinates 4 and 16, giving a diameter of the desired circle (why?)

7. (i) By the binomial theorem $(\sqrt{2}+1)^{10} = \sum_{i=0}^{10} C_i(\sqrt{2})^i$, where C_i are appropriate constants. Write the value of i for which $C_i(\sqrt{2})^i$ is the largest among the 11 terms in this sum.

   Answer: $i = 6$. One way: simplify the ratio $\frac{C_{i+1}(\sqrt{2})^{i+1}}{C_i(\sqrt{2})^i}$ and see that this ratio is > 1 till $i = 5$ and < 1 from $i = 6$ onwards.

   (ii) For every natural number n, let $(\sqrt{2}+1)^n = p_n + \sqrt{2}\, q_n$, where p_n and q_n are integers. Calculate $\lim_{n\to\infty} (\frac{p_n}{q_n})^{10}$.

   Answer: 32. Using binomial expansion see that $(\sqrt{2}-1)^n = \pm(p_n - \sqrt{2}\, q_n)$, where the sign depends on the parity of n. As $n \to \infty, (\sqrt{2}-1)^n \to 0$ since $(\sqrt{2}-1) < 1$. Thus $(p_n - \sqrt{2}\, q_n) \to 0$ and so $\frac{p_n}{q_n} \to \sqrt{2}$.

8. The format for car license plates in a small country is two digits followed by three vowels, e.g. 04 *IOU*. A license plate is called "confusing" if the digit 0 (zero) and the vowel O are both present on it. For example 04 *IOU* is confusing but 20 *AEI* is not. (i) How many distinct number plates are possible in all? (ii) How many of these are *not* confusing?

   Answer: (i) $10^2 \times 5^3 = 12500$. (ii) $10^2 \times 4^3$ plates without vowel O + $9^2 \times (5^3 - 4^3)$ plates with vowel O but without 0. This gives $6400 + 4941 = 11341$.

9. Recall that $\sin^{-1}$ is the inverse function of sin, as defined in the standard fashion. (Sometimes $\sin^{-1}$ is called arcsin.) Let $f(x) = \sin^{-1}(\sin(\pi x))$. Write the values of the following. (Some answers may involve the irrational number π. Write such answers in terms of π.)

   (i) $f(2.7)$ (ii) $f'(2.7)$ (iii) $\int_0^{2.5} f(x)\, dx$

   (iv) the smallest positive x at which $f'(x)$ does not exist.

   Answer: The graph of f is periodic with period 2. From $x = -0.5$ to $x = 0.5$ it is the line $y = \pi x$ of slope π passing through the origin and from $x = 0.5$ to $x = 1.5$ it is the line with slope $-\pi$, which crosses the X axis at $x = 1$. Using this we see that (i) $f(2.7) = \sin^{-1}(\sin(2.7\pi)) = \sin^{-1}(\sin(0.7\pi)) = \sin^{-1}(\sin(0.5\pi + 0.2\pi)) = 0.5\pi - 0.2\pi = 0.3\pi$. (ii) $f'(2.7) = -\pi$ (iii) $\int_0^{2.5} f(x)\, dx = \int_2^{2.5} f(x)\, dx = \pi/8$ and (iv) the smallest positive x at which $f'(x)$ does not exist is $x = 1/2$.

10. Answer the three questions below. To answer (i) and (ii), replace ? with exactly one of the following four options: < , = , > , not enough information to compare.

    (i) Suppose z_1, z_2 are complex numbers. One of them is in the second quadrant and the other is in the third quadrant. Then $\big||z_1| - |z_2|\big| \quad ? \quad |z_1 + z_2|$.

    (ii) Complex numbers z_1, z_2 and 0 form an equilateral triangle. Then $|z_1^2 + z_2^2| \ ? \ |z_1 z_2|$.

    (iii) Let $1, z_1, z_2, z_3, z_4, z_5, z_6, z_7$ be the complex 8-th roots of unity. Find the value of $\prod_{i=1,\dots,7}(1 - z_i)$, where the symbol Π denotes product.

Answer: (i) $\big||z_1| - |z_2|\big| < |z_1 + z_2|$. One way: using triangle inequality for $z_1 + z_2$ and $-z_2$ we get $|z_1| \leq |z_1 + z_2| + |-z_2|$ and so $|z_1| - |z_2| \leq |z_1 + z_2|$. Now we may take absolute value on the LHS because switching z_1 and z_2 keeps RHS the same. For *equality*, $z_1 + z_2$ and $-z_2$ must point in the same direction, so z_1 and z_2 must be along the same line. But they are in quadrants 2 and 3, so this cannot happen.

(ii) z_2 must be obtained by rotating z_1 by angle $\pi/3$, say in the counterclockwise direction (otherwise interchange the two). Then $z_2 = z_1 e^{\frac{\pi i}{3}}$. Then $z_1^2 + z_2^2 = z_1^2(1 + e^{\frac{2\pi i}{3}})$ and $z_1 z_2 = z_1^2 e^{\frac{\pi i}{3}}$. Now $1 + e^{\frac{2\pi i}{3}} = e^{\frac{\pi i}{3}}$ (see by calculation or picture), so we have in fact $z_1^2 + z_2^2 = z_1 z_2$.

(iii) We have $\underset{i=1,\ldots,7}{\Pi}(x - z_i) = \frac{x^8-1}{x-1} = 1 + x + \ldots + x^7$. Putting $x = 1$ gives $\underset{i=1,\ldots,7}{\Pi}(1 - z_i) = 8$.

11. There are four distinct balls labelled 1,2,3,4 and four distinct bins labelled A,B,C,D. The balls are picked up in order and placed into one of the four bins at random. Let E_i denote the event that the first i balls go into distinct bins. Calculate the following probabilities.

Notation: $Pr[X]$ = the probability of event X taking place. $Pr[X|Y]$ = the probability of event X taking place, given that event Y has taken place.

Answer:

(i) $Pr[E_4] = \frac{4!}{4^4} = \frac{3}{32}$ (ii)$Pr[E_4|E_3] = \frac{1}{4}$ (iii) $Pr[E_4|E_2] = \frac{2!}{4^2} = \frac{1}{8}$ (iv) $Pr[E_3|E_4] = 1$.

Part B Solutions

---

1. Carefully solve the following series of questions. If you cannot solve an earlier part, you may still assume the result in it to solve a later part.

(a) For any polynomial $p(t)$, the limit $\lim_{t\to\infty} \frac{p(t)}{e^t}$ is independent of the polynomial p. Justify this statement and find the value of the limit.

(b) Consider the function defined by

$$\begin{aligned} q(x) &= e^{-1/x} \text{ when } x > 0 , \\ &= 0 \text{ when } x = 0 , \\ &= e^{1/x} \text{ when } x < 0 . \end{aligned}$$

Show that $q'(0)$ exists and find its value. Why is it enough to calculate the relevant limit from only one side?

(c) Now for any positive integer n, show that $q^{(n)}(0)$ exists and find its value. Here $q(x)$ is the function in part (b) and $q^{(n)}(0)$ denotes its n-th derivative at $x = 0$.

Answer: (a) If $p(t)$ is constant, then the limit $= 0$. Otherwise we get a form $\frac{\pm\infty}{\infty}$. Using L'Hospital's rule, we get $\lim_{t\to\infty} \frac{p(t)}{e^t} = \lim_{t\to\infty} \frac{p'(t)}{e^t} = 0$ by induction on the degree of t (or apply L'Hospital's rule repeatedly).

(b) The right side derivative $= \lim_{h\to 0^+} \frac{q(h)-q(0)}{h} = \lim_{h\to 0^+} \frac{e^{-1/h}}{h} = \lim_{h\to 0^+} \frac{1/h}{e^{1/h}} = \lim_{t\to +\infty} \frac{t}{e^t}$. (Let $t = 1/h$. As $h \to 0^+, t \to +\infty$.) This limit is 0, e.g. by part (a).

Now q is an even function, so letting $k = -h$, the left side derivative $= \lim_{h\to 0^-} \frac{q(h)-q(0)}{h} = \lim_{k\to 0^+} \frac{q(-k)}{-k} = \lim_{k\to 0^+} \frac{q(k)}{-k}$. Using the earlier calculation this also equals $-0 = 0$.

Note: It is wrong to argue that $q'(0) = \lim_{x\to 0} q'(x)$ because to do so, we first need to know that q' is continuous at 0, but we have not even shown that $q'(0)$ exists! For the same reason it is wrong to argue below that $q^{(n)}(0) = \lim_{x\to 0} q^{(n)}(x)$.

(c) We will show by induction on n that $q^{(n)}(0) = 0$. The case $n = 1$ is done. (We can even start the induction at $n = 0$ by interpreting $q^{(0)}(x) = q(x)$.) Assuming that we are done up to n and to prove the statement for $n+1$, we need to calculate $\lim_{h\to 0} \frac{q^{(n)}(h)-q^{(n)}(0)}{h} = \lim_{h\to 0} \frac{q^{(n)}(h)}{h}$, because $q^{(n)}(0) = 0$ by induction. Therefore it is good to examine $q^{(n)}(h)$ for $h \neq 0$. This is easy to calculate by the usual rules, but the formulas will be different for positive and negative h. For $h \neq 0$, as q is even, q' is odd, so q'' is even, etc. and in general $q^{(n)}(h) = (-1)^n q^{(n)}(-h)$. Therefore, just as for part (b), it suffices to show that $\lim_{h\to 0^+} \frac{q^{(n)}(h)}{h} = 0$. By another induction, we see easily that for $h > 0$, $q^{(n)}(h) = p(1/h)e^{-1/h}$ for some polynomial p. $\Big[$ Proof: $q'(h) = (\frac{1}{h^2})e^{-1/h}$. Assuming the result for n, we have $q^{(n+1)}(h) = [p(1/h)e^{-1/h}]' = -\frac{1}{h^2}(-p(1/h) + p'(1/h))e^{-1/h}$, which has the desired form.$\Big]$ So we have $\lim_{h\to 0^+} \frac{q^{(n)}(h)}{h} = \lim_{h\to 0^+} \frac{p(1/h)e^{-1/h}}{h} = \lim_{t\to\infty} tp(t)e^{-t} = \lim_{t\to\infty} \frac{tp(t)}{e^t} = 0$ by part (a). Here we have again substituted $t = 1/h$.

2. Let p, q and r be real numbers with $p^2 + q^2 + r^2 = 1$.

(a) Prove the inequality $3\,p^2q + 3\,p^2r + 2\,q^3 + 2\,r^3 \;\le\; 2$.

(b) Also find the smallest possible value of $3\,p^2q + 3\,p^2r + 2\,q^3 + 2\,r^3$. Specify exactly when the smallest and the largest possible value is achieved.

Answer: We have $3\,p^2q + 3\,p^2r + 2\,q^3 + 2\,r^3 = (q+r)(3p^2 + 2q^2 + 2r^2 - 2qr) = (q+r)(3(p^2+q^2+r^2) - (q^2+r^2+2qr)) = (q+r)(3-(q+r)^2) = x(3-x^2) = 3x - x^3$, where $x = q + r$. Let us examine possible values of x in view of the constraint $p^2 + q^2 + r^2 = 1$.

We have $2qr \le q^2 + r^2$ e.g. because $(q-r)^2 \ge 0$. Adding $q^2 + r^2$, we get $q^2 + r^2 + 2qr \le 2q^2 + 2r^2 \le 2$, because $q^2 + r^2 \le p^2 + q^2 + r^2 = 1$. Thus $(q+r)^2 \le 2$. So $-\sqrt{2} \le q + r \le \sqrt{2}$. Note that equalities are achieved precisely when $p = 0$ and $q = r = \pm 1/\sqrt{2}$.

Thus altogether we have to find extrema of the odd function $f(x) = 3x - x^3$ over the interval $[-\sqrt{2}, \sqrt{2}]$. The critical points are when $f'(x) = 3 - 3x^2 = 0$, i.e. $x = \pm 1$. Thus we need to see only $f(\pm\sqrt{2}) = \pm\sqrt{2}$ and $f(\pm 1) = \pm 2$. Therefore $-2 \le\; 3\,p^2q + 3\,p^2r + 2\,q^3 + 2\,r^3 \le 2$. Moreover, $3\,p^2q + 3\,p^2r + 2\,q^3 + 2\,r^3 = \pm 2$ precisely when $x = q + r = \pm 1$. In each case, this gives a line segment in the qr-plane joining $(\pm 1, 0)$ and $(0, \pm 1)$. Note that both these segments lie within the circle $q^2 + r^2 = 1$, so each point on them leads to two valid points (p, q, r) on the unit sphere.

3. (a) Show that there are exactly 2 numbers a in $\{2, 3, \ldots, 9999\}$ for which $a^2 - a$ is divisible by 10000. Find these values of a.

(b) Let n be a positive integer. For how many numbers a in $\{2, 3, \ldots, n^2 - 1\}$ is $a^2 - a$ divisible by n^2 ? State your answer suitably in terms of n and justify.

Answer: (a) We have $10000 = 16 \times 625$ as product of prime powers. Recall the notation $a|b$, meaning b is divisible by a. We have $10000|a^2 - a$ if and only if $(625|a(a-1)$ and $16|a(a-1))$. Because a and $a-1$ cannot share a factor, in turn this is equivalent to having both the conditions (1) $625|a$ or $625|a-1$ AND (2) $16|a$ or $16|a-1$. Now if the coprime integers 16 and 625 both divide the same natural number (in our case a or $a-1$), their product 10000 will also divide this number. In our case this would force $a = 0, 1$, or ≥ 10000, all of which are not allowed. Thus the given requirement on a is equivalent to having either (1) $16|a$ and $625|a-1$ OR (2) $16|a-1$ and $625|a$. Each case has a unique solution, respectively $a = 9376$ and $a = 625$ (e.g. use modular arithmetic: in case 1, we have $a = 625k + 1$, which is $k + 1$ mod 16, forcing $k = 15$ because $16|a$ and $a \in \{2, 3, \ldots, 9999\}$).

(b) Let $n = {p_1}^{e_1} \ldots {p_k}^{e_k}$ be the factorization of n into powers of distinct primes. The analysis in part (a) tells that required values of a are obtained as follows: write $n^2 = xy$ as a product of two coprime integers and find values of a in $\{2, 3, \ldots, n^2 - 1\}$ that are simultaneously 0 *mod* x and 1 *mod* y. These are precisely the values of a that we want. This is because each $p_i^{2e_i}$ must divide a or $a-1$, as a and $a-1$ are coprime.

Now the Chinese remainder theorem tells you that there is always an a that is 0 *mod* x and 1 *mod* y. Moreover it is unique modulo $xy = n^2$ because difference between any two solutions would be divisible by xy.

The total number of ways to write $n^2 = xy$ as a product of coprime integers is exactly 2^k as it amounts to choosing which of the k distinct primes to include in x and then the rest go into y. (Notice that x and y are not interchangeable.) However, we have to delete the two cases $x = 1, y = n^2$ and $y = 1, x = n^2$, as these will respectively lead to solutions $a = 1$ and $a = 0$ or n^2, which are not in $\{2, 3, \ldots, n^2 - 1\}$. Finally it is easy to see that different choices of x lead to different values of a. This is because, of the primes $p_1, \ldots, p_k$ in the factorization of n, precisely the ones dividing x will divide a and the remaining primes will not, because they divide $a - 1$.

Thus the final answer is $2^k - 2$. Note that this matches with the special case in part (a). Finally, note that there was nothing special about taking a square: instead of n^2 it could be any positive integer m and we would proceed the same way to find requisite integers a in $\{2, 3 \ldots, m - 1\}$ based on prime factorization of m.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be a twice differentiable function, where $\mathbb{R}$ denotes the set of real numbers. Suppose that for all real numbers x and y, the function f satisfies

$$f'(x) - f'(y) \leq 3|x - y|.$$

Answer the following questions. No credit will be given without full justification.

(a) Show that for all x and y, we must have $|f(x) - f(y) - f'(y)(x - y)| \leq 1.5(x - y)^2$.

(b) Find the largest and smallest possible values for $f''(x)$ under the given conditions.

Answer: (a) Note that the given inequality stays valid if we take absolute value of the LHS, because we may interchange x and y without affecting RHS.

Fix x, y and let $t = x - y$. For now let $x \geq y$, i.e. $t \geq 0$. For $h \in [0, t]$, the value of $y + h$ varies between y and x. We are given that $|f'(y + h) - f'(y)| \leq 3|h|$. Integrate with respect to h over the interval $[0, t]$ to get $\int_0^t |f'(y + h) - f'(y)| dh \leq \int_0^t 3|h| dh = 1.5t^2$. The LHS should remind us of the following general fact: the absolute value of a definite integral $\leq$ the definite integral of absolute value the same function over the same interval. So we get $|\int_0^t f'(y + h) - f'(y) dh| \leq \int_0^t |f'(y + h) - f'(y)| dh$. Combining with the previous inequality we have $|\int_0^t f'(y + h) - f'(y) dh| \leq 1.5t^2$. Finally we calculate the LHS. $|\int_0^t f'(y + h) dh - \int_0^t f'(y) dh| = |f(y + t) - f(y) - f'(y)t|$, where the first integral is calculated using the fundamental theorem of calculus and the second one is just the integral of the constant $f'(y)$. Substituting $x - y$ for t gives the desired result.

Notes: (1) If $x < y$, then $t < 0$. We use the same strategy but all definite integrals should be taken over $[t, 0]$. Now $\int_t^0 3|h| dh = 1.5t^2 =$ LHS of the desired inequality. In the final calculation we get $\int_t^0 f'(y + h) - f'(y) dh = f(y) - f(y + t) + f'(y)t =$ negative of the previous answer. So when we take absolute value of this integral, we again get the same RHS of the desired inequality. (2) $f'(y + h) - f'(y)$ and its absolute value are integrable functions of h because they are continuous. This is because $f'(y + h)$ is just a shift of the function f' and f' is continuous because it is differentiable by hypothesis.

(b) We have, for $x \neq y$, $|\frac{f'(x)-f'(y)}{x-y}| \leq 3$, so $-3 \leq \frac{f'(x)-f'(y)}{x-y} \leq 3$. Taking limit as $y \to x$, we get $-3 \leq f''(x) \leq 3$. It is easy to provide examples where f'' attains the extreme values ± 3, e.g. $f(x) = \pm 1.5x^2$. These satisfy the hypothesis and have constant $f'' = \pm 3$.

5. For an arbitrary integer n, let $g(n)$ be the GCD of $2n + 9$ and $6n^2 + 11n - 2$. What is the largest positive integer that can be obtained as the value of $g(n)$? If $g(n)$ can be arbitrarily large, state so explicitly and prove it.

   Answer: Long division gives $6n^2 + 11n - 2 = (2n + 9)(3n - 8) + 70$. By Euclidean algorithm, $\text{GCD}(6n^2 + 11n - 2, 2n + 9) = \text{GCD}(2n + 9, 70)$. Thus $g(n)$ divides 70. But since $g(n)$ divides $2n + 9$, which is odd, $g(n)$ divides 35. When $n = 13$, $2n + 9 = 35$ and hence $g(13) = 35$. Thus the maximum value of $g(n)$ is 35. (Precisely for which n do we have $g(n) = 35$ (or, if you wish, 1 or 5 or 7)? A bit more work will tell you. Try it.)

6. You are given the following: a circle, one of its diameters AB and a point X.

   (a) Using only a straight-edge, show in the given figure how to draw a line perpendicular to AB passing through X. No credit will be given without full justification. (Recall that a straight-edge is a ruler without any markings. Given two points, a straight-edge can be used to draw the line passing through the given points.)

   Answer: Line AX cuts the circle in C. Line BX cuts the circle in D. Lines AD and BC intersect in E. Line XE is perpendicular to line AB. Reason: Angles ADB and ACB are right angles, being angles in a semicircle. The altitudes of triangle XAB are concurrent. Two of them are AD and BC, so the third is contained in line XE. (Notice that we always use lines rather than line segments - this is important for part (b).)

(b) Do NOT draw any of your work for this part in the given figure. Reconsider your procedure to see if it can be made to work if the point X is in some other position, e.g., when it is inside the circle or to the "left/right" of the circle. Clearly specify all positions of the point X for which your procedure in part (a), or a small extension/variation of it, can be used to obtain the perpendicular to AB through X. Justify your answer.

Answer: *Case 1:* Suppose X is not on the line AB (so XAB is a triangle), nor on the tangents to the circle at A (so line XA meets the circle in a point C different from A), nor on the tangent at B (so line XB meets the circle in a point D different from B) nor on the given circle (so C, D and X are all different). In this case the exact same procedure will work so long as we understand that the altitudes and their intersection point may lie outside triangle XAB. This is because the lines XA and XB meet the circle in two distinct points C and D that are different from X,A and B.

Case 2: Suppose X is on one of the two tangents, say the tangent at A, but X is different from A. In this case XA itself is the desired line! In terms of the construction, here we have A = C = E. Of course we have to assume that we can detect whether a line meeting a circle does so in one point or two. But this assumption is implicit in Case 1 also, because there we need to be able to identify the second point of intersection!

Case 3: If X is on line AB, then XAB is not a triangle. If X is not on line AB but X is on the circle, then XAB is a triangle but X = C = D = E, so we cannot draw line XE. Thus in these cases, the *above procedure* fails. **Nonetheless even in these cases it is possible to draw a perpendicular through X to line AB using only a straightedge. It is a challenge to you to find a suitable procedure!**