2015 Entrance Examination for BSc Programmes at CMI

Read the instructions on the front of the booklet carefully!

Part A. Write your final answers on page 3.

Part A is worth a total of 44 points = 4 points each for 11 problems. Points will be given based only on clearly legible final answers filled in the correct place on page 3. Write all answers for a single question on the designated line and in the order in which they are asked, separated by commas.

Unless specified otherwise, each answer is either a rational number or, where appropriate, one of the phrases "infinite" or "does not exist". If the answer is an integer, write it in the usual decimal form. Write non-integer rationals as ratios of two coprime integers.

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1. Ten people sitting around a circular table decide to donate some money for charity. You are told that the amount donated by each person was the average of the money donated by the two persons sitting adjacent to him/her. One person donated Rs. 500. Choose the correct option for each of the following two questions. Write your answers as a sequence of two letters (a/b/c/d).

   What is the total amount donated by the 10 people?

   (a) exactly Rs. 5000 (b) less than Rs. 5000 (c) more than Rs. 5000

   (d) not possible to decide among the above three options.

   What is the maximum amount donated by an individual?

   (a) exactly Rs. 500 (b) less than Rs. 500 (c) more than Rs. 500

   (d) not possible to decide among the above three options.

2. Consider all finite letter-strings formed by using only two letters A and B. We consider the usual dictionary order on these strings. See below for the formal rule with examples.

   Formal rule: To compare two strings w_1 and w_2, read them from left to right. We say "w_1 is smaller than w_2" or "$w_1 < w_2$" if the first letter in which w_1 and w_2 differ is A in w_1 and B in w_2 (for example, ABAA < ABB by looking at the third letters) or if w_2 is obtained by appending some letters at the end of w_1 (e.g. AB < ABAA).

   For each of the statements below, state if it is true or false. Write your answers as a sequence of three letters (T for True and F for False) in correct order.

   (a) Let w be an arbitrary string. There exists a unique string y satisfying both the following properties: (i) $w < y$ and (ii) there is no string x with $w < x < y$.

   (b) It is possible to give an infinite decreasing sequence of strings, i.e. a sequence $w_1, w_2, \ldots$, such that $w_{i+1} < w_i$ for each positive integer i.

   (c) Fewer than 50 strings are smaller than ABBABABB.

3. A positive integer n is called a *magic number* if it has the following property: if a and b are two positive numbers that are not coprime to n then $a+b$ is also not coprime to n. For example, 2 is a magic number, because sum of any two even numbers is also even. Which of the following are magic numbers? Write your answers as a sequence of four letters (Y for Yes and N for No) in correct order.

   (i) 129 (ii) 128 (iii) 127 (iv) 100.

4. Let A, B and C be unknown constants. Consider the function $f(x)$ defined by

$$\begin{aligned} f(x) &= Ax^2 + Bx + C \quad \text{when } x \le 0 \ , \\ &= \ln(5x+1) \quad \text{when } x > 0 \ . \end{aligned}$$

   Write the values of the constants A, B and C such that $f''(x)$, i.e., the double derivative of f, exists for all real x. If this is not possible, write "not possible". If some of the constants cannot be uniquely determined, write "not unique" for each such constant.

5. Consider the polynomial $p(x) = (x+a_1)(x+a_2)\cdots(x+a_{10})$ where a_i is a real number for each $i = 1, \ldots, 10$. Suppose all of the eleven coefficients of $p(x)$ are positive. For each of the following statements, decide if it is true or false. Write your answers as a sequence of four letters (T/F) in correct order.

   (i) The polynomial $p(x)$ must have a global minimum. (ii) Each a_i must be positive.

   (iii) All real roots of $p'(x)$ must be negative. (iv) All roots of $p'(x)$ must be real.

6. Fill in the blanks. Let C_1 be the circle with center $(-8, 0)$ and radius 6. Let C_2 be the circle with center $(8, 0)$ and radius 2. Given a point P outside both circles, let $\ell_i(P)$ be the length of a tangent segment from P to circle C_i. The locus of all points P such that $\ell_1(P) = 3\ \ell_2(P)$ is a circle with radius ___ and center at (___ , ___).

7. (i) By the binomial theorem $(\sqrt{2}+1)^{10} = \sum_{i=0}^{10} C_i(\sqrt{2})^i$, where C_i are appropriate constants. Write the value of i for which $C_i(\sqrt{2})^i$ is the largest among the 11 terms in this sum.

   (ii) For every natural number n, let $(\sqrt{2}+1)^n = p_n + \sqrt{2}\ q_n$, where p_n and q_n are integers. Calculate $\lim_{n\to\infty} (\frac{p_n}{q_n})^{10}$.

8. The format for car license plates in a small country is two digits followed by three vowels, e.g. 04 *IOU*. A license plate is called "confusing" if the digit 0 (zero) and the vowel O are both present on it. For example 04 *IOU* is confusing but 20 *AEI* is not. (i) How many distinct number plates are possible in all? (ii) How many of these are *not* confusing?

9. Recall that $\sin^{-1}$ is the inverse function of sin, as defined in the standard fashion. (Sometimes $\sin^{-1}$ is called arcsin.) Let $f(x) = \sin^{-1}(\sin(\pi x))$. Write the values of the following. (Some answers may involve the irrational number π. Write such answers in terms of π.)

   (i) $f(2.7)$ (ii) $f'(2.7)$ (iii) $\int_0^{2.5} f(x)\ dx$

   (iv) the smallest positive x at which $f'(x)$ does not exist.

10. Answer the three questions below. To answer (i) and (ii), replace ? with exactly one of the following four options: $<$, $=$, $>$, not enough information to compare.

    (i) Suppose z_1, z_2 are complex numbers. One of them is in the second quadrant and the other is in the third quadrant. Then $\big||z_1| - |z_2|\big| \quad ? \quad |z_1 + z_2|$.

    (ii) Complex numbers z_1, z_2 and 0 form an equilateral triangle. Then $|z_1^2 + z_2^2| \;\; ? \;\; |z_1 z_2|$.

    (iii) Let $1, z_1, z_2, z_3, z_4, z_5, z_6, z_7$ be the complex 8-th roots of unity. Find the value of $\prod_{i=1,\ldots,7} (1 - z_i)$, where the symbol Π denotes product.

11. There are four distinct balls labelled 1,2,3,4 and four distinct bins labelled A,B,C,D. The balls are picked up in order and placed into one of the four bins at random. Let E_i denote the event that the first i balls go into distinct bins. Calculate the following probabilities.

    (i) $Pr[E_4]$ (ii)$Pr[E_4|E_3]$ (iii) $Pr[E_4|E_2]$ (iv) $Pr[E_3|E_4]$.

    Notation: $Pr[X] =$ the probability of event X taking place. $Pr[X|Y] =$ the probability of event X taking place, given that event Y has taken place.

Answers to part A

This is the only place that will be seen for grading part A. So carefully and clearly write the answers to each question on the designated line below. Write only the final answers, do not show any intermediate work. Illegible/unclear answers will not be considered.

A1. __________ A2. __________ A3. __________

A4. ______________________________

A5. ________________ A6. ________________

A7. ________________ A8. ________________

A9. ______________________________

A10. ______________________________

A11. ______________________________

Part B. Write complete solutions for these problems from page 6 onwards.

Part B is worth a total of 81 points = 15 points each for the first three problems + 12 points each for the last three problems. Solve these problems in the space provided for each problem from page 6. You may solve only part of a problem and get partial credit. **Clearly explain your entire reasoning. No credit will be given without reasoning.**

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1. Carefully solve the following series of questions. If you cannot solve an earlier part, you may still assume the result in it to solve a later part.

   (a) For any polynomial $p(t)$, the limit $\lim_{t\to\infty} \frac{p(t)}{e^t}$ is independent of the polynomial p. Justify this statement and find the value of the limit.

   (b) Consider the function defined by

   $$\begin{aligned} q(x) &= e^{-1/x} \ \text{ when } x > 0\ , \\ &= 0 \ \text{ when } x = 0\ , \\ &= e^{1/x} \ \text{ when } x < 0\ . \end{aligned}$$

   Show that $q'(0)$ exists and find its value. Why is it enough to calculate the relevant limit from only one side?

   (c) Now for any positive integer n, show that $q^{(n)}(0)$ exists and find its value. Here $q(x)$ is the function in part (b) and $q^{(n)}(0)$ denotes its n-th derivative at $x = 0$.

2. Let p, q and r be real numbers with $p^2 + q^2 + r^2 = 1$.

   (a) Prove the inequality $3\,p^2q + 3\,p^2r + 2\,q^3 + 2\,r^3 \ \leq\ 2$.

   (b) Also find the smallest possible value of $3\,p^2q + 3\,p^2r + 2\,q^3 + 2\,r^3$. Specify exactly when the smallest and the largest possible value is achieved.

3. (a) Show that there are exactly 2 numbers a in $\{2, 3, \ldots, 9999\}$ for which $a^2 - a$ is divisible by 10000. Find these values of a.

   (b) Let n be a positive integer. For how many numbers a in $\{2, 3, \ldots, n^2 - 1\}$ is $a^2 - a$ divisible by n^2 ? State your answer suitably in terms of n and justify.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be a twice differentiable function, where $\mathbb{R}$ denotes the set of real numbers. Suppose that for all real numbers x and y, the function f satisfies

   $$f'(x) - f'(y) \leq 3|x - y|.$$

   Answer the following questions. No credit will be given without full justification.

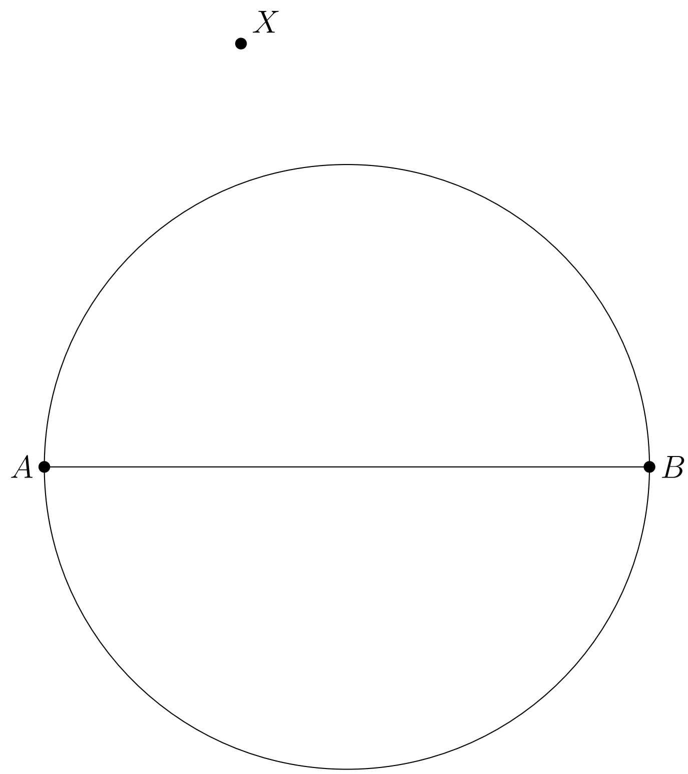
   (a) Show that for all x and y, we must have $|f(x) - f(y) - f'(y)(x - y)| \leq 1.5(x - y)^2$.

   (b) Find the largest and smallest possible values for $f''(x)$ under the given conditions.

5. For an arbitrary integer n, let $g(n)$ be the GCD of $2n + 9$ and $6n^2 + 11n - 2$. What is the largest positive integer that can be obtained as the value of $g(n)$? If $g(n)$ can be arbitrarily large, state so explicitly and prove it.

6. You are given the following: a circle, one of its diameters AB and a point X.

(a) Using only a straight-edge, show in the given figure how to draw a line perpendicular to AB passing through X. No credit will be given without full justification. (Recall that a straight-edge is a ruler without any markings. Given two points, a straight-edge can be used to draw the line passing through the given points.)



(b) Do NOT draw any of your work for this part in the given figure. Reconsider your procedure to see if it can be made to work if the point X is in some other position, e.g., when it is inside the circle or to the "left/right" of the circle. Clearly specify all positions of the point X for which your procedure in part (a), or a small extension/variation of it, can be used to obtain the perpendicular to AB through X. Justify your answer.

Write answers to part B from the next page.

1. Carefully solve the following series of questions. If you cannot solve an earlier part, you may still assume the result in it to solve a later part.

   (a) For any polynomial $p(t)$, the limit $\lim_{t\to\infty} \frac{p(t)}{e^t}$ is independent of the polynomial p. Justify this statement and find the value of the limit.

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   If you need extra space for this or any problem, continue on one of the later blank pages and write a note to that effect.

2. Let p, q and r be real numbers with $p^2 + q^2 + r^2 = 1$.

   (a) Prove the inequality $3\,p^2q + 3\,p^2r + 2\,q^3 + 2\,r^3 \;\leq\; 2$.

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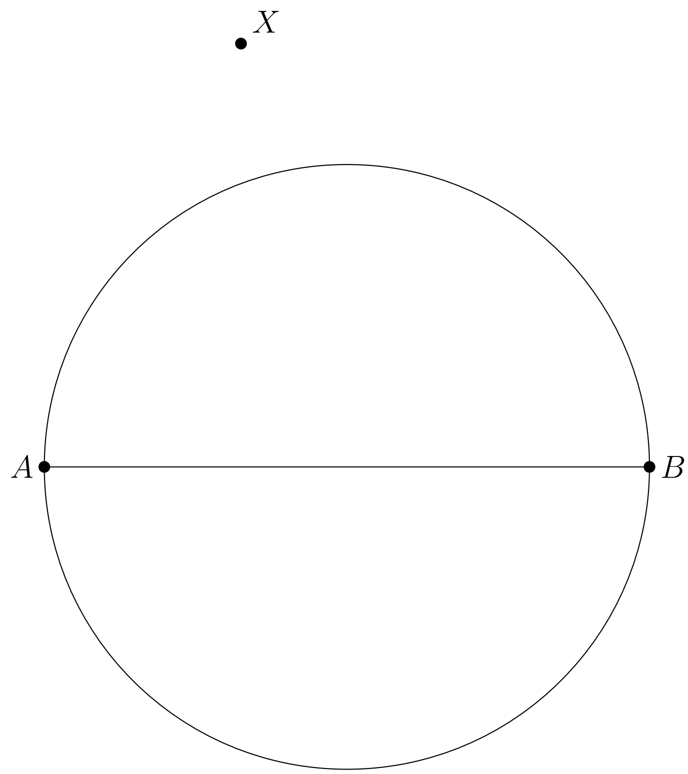

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