Solutions to 2016 Entrance Examination for BSc Programmes at CMI

Answers to Part A

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1. If K comes second, then L was third (one correct answer for R). But then R would also need to be second (one correct answer for M), a contradiction. So K cannot be second. So M must have won, etc. The order is M R L K.

2. Per-capita GDP is $\frac{\text{GDP}}{\text{population}}$. Letting G and P denote the old GDP and population respectively, the new per-capita GDP is $\frac{1.078G}{(1+x)P}$ where x is the unknown percent change in population we wish to calculate. The percent increase in per-capita GDP is $10\% = 0.1$. So we have $\frac{1.078}{1+x} = 1.1$. Solving for x we get $1+x = \frac{1.078}{1.1} = \frac{98\times 11}{100\times 11} = 0.98$. So x is -0.02. So population decreased by 2%.

3. Given $n = pq = 110179$. The number of integers relatively prime to n and smaller than n is $(p-1)(q-1)$. So we have $pq - p - q + 1 = 109480$. We get $p + q = 700$. Now p, q are solutions to the quadratic $x^2 - 700x + 110179$. The discriminant of this quadratic is $\sqrt{490000 - 440716} = \sqrt{49284} = 22$. So we get $p = \frac{700+222}{2} = 461$ and $q = \frac{700-222}{2} = 239$.

4. Let there be a steps to the right (east), b steps north-west and c steps southwest. The total number of steps is $a+b+c$. The key idea is to think of the northwest step as a move in the complex plane along ω, the complex cube root of unity, the southwest step as a move in the complex plane along ω^2 and the step to the right as a move along $\omega^3 = 1$. From the hypothesis we then have $a + b\omega + c\omega^2 = 1$. Using $1 + \omega + \omega^2 = 0$ we see that $a - 1 = b = c$. This then rules out $a + b + c = 6$, so the number of 6 step paths is zero. A 7 step path is possible only with $a = 3, b = 2, c = 2$. The number of such paths is the multinomial coefficient $\binom{7}{3,2,2} = 210$. (Instead of complex numbers one can also think in terms of vector addition in the plane.)

5. Let $\theta = \frac{\pi}{101}$. Let $A = \cos^2(\theta) + \cos^2(2\theta) + \cdots + \cos^2(100\theta)$. Let $B = \sin^2(\theta) + \sin^2(2\theta) + \cdots + \sin^2(100\theta)$. We have $A + B = 100$, and $A - B = \cos(2\theta) + \cos(4\theta) + \cdots + \cos(200\theta)$. Since $\theta = \frac{\pi}{101}$, we see that $\cos(2\theta) = \cos(\frac{2\pi}{101})$ is the real part of $e^{\frac{2\pi i}{101}}$, i being the complex square root of -1. Interpreting the other terms in $A - B$ similarly we see that $A - B$ is the real part of the sum of the 101^{th} roots of unity except 1. So $A - B = -1$. This gives $A = \frac{99}{2}, B = \frac{101}{2}$.

6. The given function is defined using the two functions $x^2 + 1$ and $\tan(x)$. Both these functions are continuous wherever they are defined. Since every irrational number z has a non terminating, non repeating decimal expansion we see that given any $\epsilon > 0$ there is a rational number p such that the distance between z and p is less than ϵ. Using these facts one can see that the given function will be continuous precisely at those x in the interval $[0, 4\pi]$ where $x^2 + 1 = \tan(x)$. Since $x^2 + 1$ is positive, it will intersect $\tan(x)$ exactly once in the intervals $[0, \frac{\pi}{2}], [\pi, \frac{3\pi}{2}], [2\pi, \frac{5\pi}{2}], [3\pi, \frac{7\pi}{2}]$, as $\tan(x)$ increases from 0 to ∞ in each of these intervals. $\tan(x)$ is negative elsewhere in the given domain. So we have 4 points of continuity.

7. TTFF

   Since the set S is nonempty, there is an element $m \in S$. But then $m = m + 0$ and so $0 \in S$. 1 cannot be in S, otherwise it will contain all non-negative integers. It is not difficult to see by the division algorithm that if m, n are in S then so is their GCD. Therefore two coprime numbers cannot be in S. Otherwise their GCD, which is 1, will be in S, a contradiction. It follows that such sets S are precisely those of the form $n\mathbb{Z}_{\geq 0}$, the set of all non-negative multiples of a fixed non-negative integer n. So there are infinitely many such possible sets.

8. TTFF

   If $g(x)$ is linear, it is $3x + 5$ because the values at 1 and 2 are 8 and 11 respectively. If $g(x)$ is a polynomial then it is $3x + 5$ plus a multiple of $(x-1)(x-2)\cdots(x-15)$. So $g(x)$ cannot be a polynomial of degree 10. But it can be a polynomial of degree 15 or more. g being differentiable does not mean that it is a polynomial. You can fit any number of differentiable functions to the given data.

9. TFFT

   i The mean value theorem tells us $S \subset T$.

   ii $T \subset S$ is false, example $f(x) = \sin(x)$. Here $f'(0) = 1$ is in T but not in S.

   iii $T = S = \mathbb{R}$ can happen at points where f is not differentiable.

   iv S has mean value property, because of continuity. (Why?)

10. TFFT

    BP and CP are angle bisectors meeting at P, so AP bisects $\angle A$ since the angle bisectors are concurrent. The angles marked with symbol $\circ$ at point P are all $60°$ because $\angle EPD =$ twice this common value. It follows that half the sum of $\angle B$ and $\angle C$ is $60°$. So $\angle A$ is $60°$. The others are false, in fact check that any triangle with $\angle A = 60°$, angle bisectors BD and CE, their point of intersection P and PF bisecting $\angle BPC$ will satisfy the given data. All four statements are true if and only if the triangle ABC is equilateral.

Solutions to Part B.

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1. Out of the 14 students taking a test, 5 are well prepared, 6 are adequately prepared and 3 are poorly prepared. There are 10 questions on the test paper. A well prepared student can answer 9 questions correctly, an adequately prepared student can answer 6 questions correctly and a poorly prepared student can answer only 3 questions correctly.

   For each probability below, write your final answer as a rational number in lowest form.

   (a) If a randomly chosen student is asked two distinct randomly chosen questions from the test, what is the probability that the student will answer both questions correctly?

Note: The student and the questions are chosen independently of each other. "Random" means that each individual student/each pair of questions is equally likely to be chosen.

(b) Now suppose that a student *was* chosen at random and asked two randomly chosen questions from the exam, and moreover *did* answer both questions correctly. Find the probability that the chosen student was well prepared.

Solution. (a) The probability that a randomly chosen student is well prepared is 5/14. The probability of a well prepared student answering two randomly chosen questions correctly is $\binom{9}{2}/\binom{10}{2}$. So the probability that a randomly chosen student is well prepared AND answers two randomly chosen questions correctly is $\frac{5}{14} \times \frac{\binom{9}{2}}{\binom{10}{2}} = \frac{2}{7}$. A student belongs to exactly one of the three preparedness categories, so the desired probability is obtained by adding $\frac{2}{7}$ with the results of parallel calculations for the other two categories. We get

$P(\text{both answers correct}) =$

$$P(\text{well prepared})\frac{\binom{9}{2}}{\binom{10}{2}} + P(\text{moderately prepared})\frac{\binom{6}{2}}{\binom{10}{2}} + P(\text{weakly prepared})\frac{\binom{3}{2}}{\binom{10}{2}},$$

which equals

$$\frac{5}{14} \times \frac{36}{45} + \frac{6}{14} \times \frac{15}{45} + \frac{3}{14} \times \frac{3}{45} = \frac{31}{70}.$$

(b) The probability that a randomly chosen student was well prepared given that he answered both questions correctly is

$$P(\text{well prepared}|\text{both correct}) = \frac{P(\text{well prepared and both correct})}{P(\text{both correct})} = \frac{2/7}{31/70} = \frac{20}{31}.$$

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2. By definition the region *inside* the parabola $y = x^2$ is the set of points (a, b) such that $b \geq a^2$. We are interested in those circles all of whose points are in this region. A *bubble at a point* P on the graph of $y = x^2$ is the *largest* such circle that contains P. (You may assume the fact that there is a *unique* such circle at any given point on the parabola.)

   (a) A bubble at some point on the parabola has radius 1. Find the center of this bubble.

   (b) Find the radius of the smallest possible bubble at some point on the parabola. Justify.

Solution. A bubble at the point $P = (a, a^2)$ must be tangential to the parabola at (a, a^2). (Why?) It must also be symmetric with respect to Y-axis (why?) and so its center O must be on the Y-axis. The radius OP of this bubble is perpendicular to the common tangent to the parabola and to the bubble at P. The slope of this tangent = $2a$, so the slope of radius $OP = \frac{-1}{2a}$ (for $a \neq 0$). Let $Q = (0, a^2)$. Using triangle OPQ, slope of $OP = \frac{-OQ}{a} = \frac{-1}{2a}$. Therefore $OQ = \frac{1}{2}$, regardless of the value of a.

(a) By Pythagoras, $OP^2 = (\frac{1}{2})^2 + a^2 = 1$. So $a^2 = \frac{3}{4}$ and $P = (0, \frac{3}{4} + \frac{1}{2}) = (0, \frac{5}{4})$.

(b) For any nonzero a, the radius of the bubble satisfies $OP^2 = (\frac{1}{2})^2 + a^2$, so $OP > \frac{1}{2}$. The smallest bubble is at the origin and its radius is $\frac{1}{2}$. (One cannot just directly take $a = 0$ in the above calculations. Argue by continuity or do a separate calculation at the origin.)

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3. Consider the function $f(x) = x^{\cos(x)+\sin(x)}$ defined for $x \geq 0$.

(a) Prove that

$$0.4 \leq \int_0^1 f(x)dx \leq 0.5.$$

Solution. It is easy to see that for $0 \leq x \leq 1$, we have $1 \leq \cos(x) + \sin(x) \leq \sqrt{2}$, and so

$$x^1 \geq x^{\cos(x)+\sin(x)} \geq x^{\sqrt{2}}.$$

As all three functions are non-negative in [0,1], we can integrate the inequalities over that interval to get

$$\frac{1}{2} \geq \int_0^1 f(x)dx \geq \frac{1}{\sqrt{2}+1} > \frac{1}{1.5+1} = 0.4.$$

(b) Suppose the graph of $f(x)$ is being traced on a computer screen with the uniform speed of 1 cm per second (i.e., this is how fast the length of the curve is increasing). Show that at the moment the point corresponding to $x = 1$ is being drawn, the x coordinate is increasing at the rate of

$$\frac{1}{\sqrt{2+\sin(2)}} \text{ cm per second.}$$

Solution. Length of the curve from $x = 0$ to any given x is $l(x) = \int_0^x \sqrt{1+f'(u)^2}du$. It is given that $\frac{dl}{dt} = 1$ cm/second at all times. One needs to find $\frac{dx}{dt}$ when $x = 1$.

By chain rule $\frac{dl}{dt} = \frac{dl}{dx}\frac{dx}{dt}$. By the fundamental theorem of calculus $\frac{dl}{dx} = \sqrt{1+f'(x)^2}$. We calculate $f'(1) = \cos(1) + \sin(1)$. (Use $f(x) = x^{\cos(x)+\sin(x)} = e^{\ln x(\cos(x)+\sin(x))}$, etc.) So at $x = 1$, $\frac{dl}{dx} = \sqrt{1+(\cos(1)+\sin(1))^2} = \sqrt{2+\sin 2}$. Chain rule gives the answer.

(Remark: We are using calculus to analyze what in reality is a discrete situation, as a computer will draw pixel by pixel. So the whole description is an approximation. It is also probably more realistic to assume $\frac{dx}{dt}$ to be constant.)

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4. Let A be a non-empty finite sequence of n distinct integers $a_1 < a_2 < \cdots < a_n$. Define

$$A + A = \{a_i + a_j | 1 \leq i, j \leq n\},$$

i.e., the set of all pairwise sums of numbers from A. E.g., for $A = \{1, 4\}$, $A+A = \{2, 5, 8\}$.

(a) Show that $|A + A| \geq 2n - 1$. Here $|A + A|$ means the number of elements in $A + A$.

(b) Prove that $|A+A| = 2n-1$ if and only if the sequence A is an arithmetic progression.

(c) Find a sequence A of the form $0 < 1 < a_3 < \cdots < a_{10}$ such that $|A + A| = 20$.

Solution. (a) Easy induction, see answer to (b). Or explicitly, one has the $2n-1$ distinct numbers $a_1 + a_1 < a_1 + a_2 < \cdots < a_1 + a_n < a_2 + a_n < \ldots < a_n + a_n$ in $A + A$. (A way to visualize is to write $a_i + a_j$ at point (i, j) in the XY-plane. Any step to the right or up increases the number. To reach from $2a_1$ to $2a_n$ needs $2n - 1$ such steps. The given example is the path along bottom row and then rightmost column.)

(b) Suppose the a_i form an arithmetic progression. Then for a fixed k, the value of $a_i + a_{k-i}$ is constant for all possible i, where $2 \leq k \leq 2n$. For the converse use induction. There is nothing to prove for $n = 1, 2$. For $n > 2$, remove a_n from A to get a set B. Now $|A + A| - |B + B| \geq 2$, because the two distinct numbers $a_{n-1} + a_n$ and $2a_n$ in $A + A$ are greater than all numbers in $B + B$. So for $|A + A| = 2n - 1$ to happen, one must have $|B + B| = 2n - 3$, which by induction forces $a_1, \ldots, a_{n-1}$ to be in an arithmetic progression. Moreover $a_{n-2} + a_n$ must be in $B + B$ and it can only be the largest number $2a_{n-1}$ (because all others are smaller than $a_{n-2} + a_n$). This shows that a_n is the next term of the same arithmetic progression.

(c) 0,1,2,3,4,5,6,7,8,10. This answer is unique. (Why?)

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5. Find a polynomial $p(x)$ that simultaneously has both the following properties.

(i) When $p(x)$ is divided by x^{100} the remainder is the constant polynomial 1.

(ii) When $p(x)$ is divided by $(x - 2)^3$ the remainder is the constant polynomial 2.

Solution. Suppose a polynomial $f(x)$ leaves a *constant* remainder r when divided by the polynomial $(x - c)^k$. Then $f'(x)$ is divisible by $(x - c)^{k-1}$. The converse is also true: suppose for a polynomial $f(x)$, the derivative $f'(x)$ is divisible by $(x - c)^{k-1}$, say $f'(x) = q(x)(x - c)^{k-1}$. Then $f(x)$ leaves a constant remainder when divided by $(x - c)^k$. One can see this e.g. by substituting $u = (x - c)$ in $q(x)(x - c)^{k-1}$ and integrating.

In the given problem $p'(x)$ must be divisible by x^{99} as well as by $(x - 2)^2$. Moreover any polynomial whose derivative is divisible by $x^{99}(x - 2)^2$ will leave constant remainders when divided by either of x^{100} and $(x - 2)^3$. The simplest way to find one such $p(x)$ is to integrate $Ax^{99}(x - 2)^2 = A(x^{101} - 4x^{100} + 4x^{99})$ to get

$$p(x) = A\left(\frac{x^{102}}{102} - \frac{4x^{101}}{101} + \frac{4x^{100}}{100}\right) + B$$

and solve for constants A and B to ensure desired values of the constant remainders. We have $p(0) = B = 1$ and $p(2) = A\left(\frac{2^{102}}{102} - \frac{4\times 2^{101}}{101} + \frac{4\times 2^{100}}{100}\right) + 1 = 2$, which gives A.

Theoretical approach. Working through the following reasoning will be very useful for your understanding of basic arithmetic/algebra. It explains how to *implememt* the Chinese remainder theorem using the Euclidean algorithm for finding GCD. This theorem states the following. *One can always find an integer that leaves desired remainders when divided by two coprime integers a and b.*

Suppose we are required to find an integer that leaves remainder r when divided by a and remainder s when divided by b. A way to achieve this systematically is to use the Euclidean algorithm, which finds GCD of two numbers by repeated division with remainder. This algorithm also enables one to write the GCD in the form $xa + yb$, where the integers x, y can be found explicitly by backward substitution in the steps used to calculate the GCD. If a and b are coprime, i.e. if their GCD is 1, then we can write $1 = xa + yb$. This tells you that xa is 1 modulo b and yb is 1 modulo a. Therefore, $sxa + ryb$ is r modulo a and s modulo b.

The relevance for this problem is that the same reasoning applies for polynomials in one variable, because in this setting too one has division with remainder. Because x^{100} and $(x-2)^3$ do not share a common factor, you know without any work that a polynomial with given properties must exist. The same algorithm as the previous paragraph (but now with polynomials) gives a systematic way to find it. In the given problem we could use a different trick because the specified remainders here were rather simple (constants). But there is a conceptual way as well by implementing the Chinese remainder theorem.

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6. Find all pairs (p, n) of positive integers where p is a prime number and $p^3 - p = n^7 - n^3$.

   Solution. The given equation is $p(p-1)(p+1) = n^3(n^2+1)(n+1)(n-1)$. As the factor p on the LHS is a prime, it must divide one of the factors $n-1, n, n+1, n^2+1$ on the RHS.

   A key point to deduce is that $p > n^2$. One way to do this is as follows. The LHS $= p^3 - p$ is an increasing function of p for $p \geq 1$, e.g. because the derivative $3p^2 - 1$ is positive. So for any given $n \geq 1$, there is exactly one real value of p for which LHS = RHS. Trying $p = n^2$ gives LHS $= n^6 - n^2 < n^7 - n^3 =$ RHS, e.g. because $n^7 - n^3 - (n^6 - n^2) = (n^6 - n^2)(n-1) > 0$.

   As the prime p is greater than n^2, it cannot divide any of $n-1, n, n+1$. So p must divide n^2+1 and therefore we must have $p = n^2 + 1$, again because $p > n^2$. Substituting this in the given equation, we get $(n^2+1)n^2(n^2+2) = n^3(n^2+1)(n+1)(n-1)$. Canceling common factors gives $n^2 + 2 = n^3 - n$, i.e. $2 = n^3 - n^2 - n$. This has a unique integer solution $n = 2$, e.g. because the factor n on the RHS must divide 2 and now one checks that $n = 2$ works. So $n = 2$ and the prime $p = n^2 + 1 = 5$ give a unique solution to the given equation.