2016 Entrance Examination for BSc Programmes at CMI

Read the instructions on the front of the booklet carefully!

Part A. Write your final answers on page 3.

Part A is worth a total of (4 × 10 = 40) points. Points will be given based only on clearly legible final answers filled in the correct place on page 3. Write all answers for a single question on the designated line and in the order in which they are asked, separated by commas.

Unless specified otherwise, each answer is either a rational number or, where appropriate, one of the phrases "infinite"/"does not exist"/"not possible to decide". Write integer answers in the usual decimal form. Write non-integer rationals as ratios of two coprime integers.

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1. Four children K, L, M and R are about to run a race. They make some predictions as follows.

   K says: M will win. Myself will come second.

   R says: M will come second. L will be third.

   M says: L will be last. R will be second.

   After the race, it turns out that each person has made exactly one correct and one incorrect prediction. Write the result of the race in the order from first to the last.

2. A country's GDP grew by 7.8% within a period. During the same period the country's per-capita-GDP (= ratio of GDP to the total population) increased by 10%. During this period, the total population of the country *increased/decreased* by _____ %. (Choose the correct option and fill in the blank if possible.)

3. You are told that $n = 110179$ is the product of two primes p and q. The number of positive integers less than n that are relatively prime to n (i.e. those m such that $gcd(m, n) = 1$) is 109480. Write the value of $p + q$. Then write the values of p and q.

4. A *step* starting at a point P in the XY-plane consists of moving by *one unit* from P in one of three directions: directly to the right or in the direction of one of the two rays that make the angle of $\pm 120°$ with positive X-axis. (An opposite move, i.e. to the left/southeast/northeast, is not allowed.) A *path* consists of a number of such steps, each new step starting where the previous step ended. Points and steps in a path may repeat.

   Find the number of paths starting at (1,0) and ending at (2,0) that consist of

   (i) exactly 6 steps (ii) exactly 7 steps.

5. Find the value of the following sum of 100 terms. (Possible hint: also consider the same sum with $\sin^2$ instead of $\cos^2$.)

$$\cos^2\left(\frac{\pi}{101}\right) + \cos^2\left(\frac{2\pi}{101}\right) + \cos^2\left(\frac{3\pi}{101}\right) + \cdots + \cos^2\left(\frac{99\pi}{101}\right) + \cos^2\left(\frac{100\pi}{101}\right).$$

6. A function $f(x)$ is defined by the following formulas

$$f(x) = \begin{cases} x^2 + 1 & \text{when } x \text{ is irrational,} \\ \tan(x) & \text{when } x \text{ is rational.} \end{cases}$$

At how many x in the interval $[0, 4\pi]$ is $f(x)$ continuous?

In each question below, four statements are given. For each statement, state if it is true or false. Write your answer to each question as a sequence of four letters (T for True and F for False) in correct order.

7. We want to construct a *nonempty* and *proper* subset S of the set of non-negative integers. This set must have the following properties. For any m and any n,

   if $m \in S$ and $n \in S$ then $m + n \in S$ and if $m \in S$ and $m + n \in S$ then $n \in S$.

   (i) 0 must be in S.

   (ii) 1 cannot be in S.

   (iii) There are only finitely many ways to construct such a subset S.

   (iv) There is such a subset S that contains both 2015^{2016} and 2016^{2015}.

8. A function g satisfies the property that $g(k) = 3k + 5$ for each of the 15 integer values of k in [1,15].

   (i) If $g(x)$ is a linear polynomial, then $g(x) = 3x + 5$.

   (ii) g cannot be a polynomial of degree 10.

   (iii) g cannot be a polynomial of degree 20.

   (iv) If g is differentiable, then g must be a polynomial.

9. Given a *continuous* function f, define the following subsets of the set $\mathbb{R}$ of real numbers.

   T = set of slopes of all possible tangents to the graph of f.

   S = set of slopes of all possible secants, i.e. lines joining two points on the graph of f.

   (i) If f is differentiable, then $S \subset T$.

   (ii) If f is differentiable, then $T \subset S$.

   (iii) If $T = S = \mathbb{R}$, then f must be differentiable everywhere.

   (iv) Suppose 0 and 1 are in S. Then every number between 0 and 1 must also be in S.

10. You are given a triangle ABC, a point D on segment AC, a point E on segment AB and a point F on segment BC. Let BD and CE intersect in point P. Join P with F. Suppose that $\angle$ EPB = $\angle$ BPF = $\angle$ FPC = $\angle$ CPD and PD = PE = PF. (See an *indicative* figure on page 3. It may not be the only such figure, so measuring it may be misleading.)

    (i) AP must bisect $\angle$ BAC.

    (ii) $\triangle$ ABC must be isosceles.

    (iii) A, P, F must be collinear.

    (iv) $\angle$ BAC must be 60°.

Answers to part A

This is the only place that will be seen for grading part A. So carefully and clearly write the answers to each question on the designated line below. Write only the final answers, do not show any intermediate work. Illegible/unclear answers will not be considered.

A1. ______________________________

A2. ______________________________

A3. ______________________________

A4. ______________________________

A5. ______________________________

A6. ______________________________

A7. ______________________________

A8. ______________________________

A9. ______________________________

A10. ______________________________

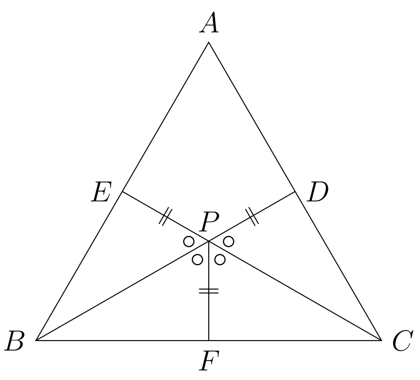

Figure for question A10

Part B. Write complete solutions for these questions from page 6 onwards.

Part B is worth a total of (6 × 14 = 84) points. Solve these questions in the space provided for each question from page 6. You may solve only part of a question and get partial credit. **Clearly explain your entire reasoning. No credit will be given without reasoning.**

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1. Out of the 14 students taking a test, 5 are well prepared, 6 are adequately prepared and 3 are poorly prepared. There are 10 questions on the test paper. A well prepared student can answer 9 questions correctly, an adequately prepared student can answer 6 questions correctly and a poorly prepared student can answer only 3 questions correctly.

   For each probability below, write your final answer as a rational number in lowest form.

   (a) If a randomly chosen student is asked two distinct randomly chosen questions from the test, what is the probability that the student will answer both questions correctly?

   Note: The student and the questions are chosen independently of each other. "Random" means that each individual student/each pair of questions is equally likely to be chosen.

   (b) Now suppose that a student *was* chosen at random and asked two randomly chosen questions from the exam, and moreover *did* answer both questions correctly. Find the probability that the chosen student was well prepared.

2. By definition the region *inside* the parabola $y = x^2$ is the set of points (a, b) such that $b \geq a^2$. We are interested in those circles all of whose points are in this region. A *bubble at a point* P on the graph of $y = x^2$ is the *largest* such circle that contains P. (You may assume the fact that there is a *unique* such circle at any given point on the parabola.)

   (a) A bubble at some point on the parabola has radius 1. Find the center of this bubble.

   (b) Find the radius of the smallest possible bubble at some point on the parabola. Justify.

3. Consider the function $f(x) = x^{\cos(x)+\sin(x)}$ defined for $x \geq 0$.

   (a) Prove that
   $$0.4 \leq \int_0^1 f(x)dx \leq 0.5.$$

   (b) Suppose the graph of $f(x)$ is being traced on a computer screen with the uniform speed of 1 cm per second (i.e., this is how fast the length of the curve is increasing). Show that at the moment the point corresponding to $x = 1$ is being drawn, the x coordinate is increasing at the rate of
   $$\frac{1}{\sqrt{2 + \sin(2)}} \text{ cm per second.}$$

4. Let A be a non-empty finite sequence of n distinct integers $a_1 < a_2 < \cdots < a_n$. Define

$$A + A = \{a_i + a_j | 1 \leq i, j \leq n\},$$

i.e., the set of all pairwise sums of numbers from A. E.g., for $A = \{1, 4\}$, $A+A = \{2, 5, 8\}$.

(a) Show that $|A + A| \geq 2n - 1$. Here $|A + A|$ means the number of elements in $A + A$.

(b) Prove that $|A+A| = 2n-1$ if and only if the sequence A is an arithmetic progression.

(c) Find a sequence A of the form $0 < 1 < a_3 < \cdots < a_{10}$ such that $|A + A| = 20$.

5. Find a polynomial $p(x)$ that simultaneously has both the following properties.

(i) When $p(x)$ is divided by x^{100} the remainder is the constant polynomial 1.

(ii) When $p(x)$ is divided by $(x - 2)^3$ the remainder is the constant polynomial 2.

6. Find all pairs (p, n) of positive integers where p is a prime number and $p^3 - p = n^7 - n^3$.

Write answers to part B from the next page.