2018 Entrance Examination for the BSc Programmes at CMI

Read the instructions on the front of the booklet carefully!

Part A. Write your final answers on page 3.

Part A is worth a total of (4 × 10 = 40) points. Points will be given based only on clearly legible final answers filled in the correct place on page 3. Write all answers for a single question on the designated line and in the order in which they are asked, separated by commas.

Unless specified otherwise, each answer is either a number (rational/ real/ complex) or, where appropriate, one of the phrases "infinite"/"does not exist"/"not possible to decide". Write integer answers in the usual decimal form. Write non-integer rationals as ratios of two integers.

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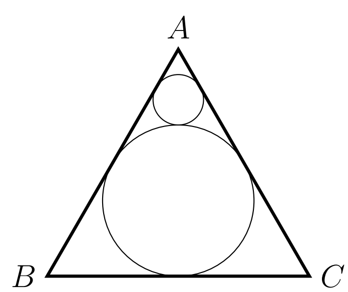
1. Consider an equilateral triangle ABC with *altitude* 3 centimeters. A circle is inscribed in this triangle, then another circle is drawn such that it is tangent to the inscribed circle and the sides AB, AC. Infinitely many such circles are drawn; each tangent to the previous circle and the sides AB, AC. The figure shows the construction after 2 steps.



Find the sum of the areas of all these circles.

2. Consider the following function defined for all real numbers x

$$f(x) = \frac{2018}{100 + e^x}.$$

How many integers are there in the range of f?

3. List every solution of the following equation. You need not simplify your answer(s).

$$\sqrt[3]{x+4} - \sqrt[3]{x} = 1.$$

4. Compute the following integral

$$\int_0^{\frac{\pi}{2}} \frac{\mathrm{d}x}{(\sqrt{\sin x} + \sqrt{\cos x})^4}.$$

5. List in increasing order all positive integers $n \leq 40$ such that n cannot be written in the form $a^2 - b^2$, where a and b are positive integers.

6. Consider the equation
$$z^{2018} = 2018^{2018} + i,$$
where $i = \sqrt{-1}$.

   (a) How many complex solutions does this equation have?

   (b) How many solutions lie in the first quadrant?

   (c) How many solutions lie in the second quadrant?

7. Let $x^3 + ax^2 + bx + 8 = 0$ be a cubic equation with integer coefficients. Suppose both r and $-r$ are roots of this equation, where $r > 0$ is a real number. List all possible pairs of values (a, b).

8. How many non-congruent triangles are there with *integer* lengths $a \leq b \leq c$ such that $a + b + c = 20$?

9. Consider a sequence of polynomials with real coefficients defined by
$$p_0(x) = (x^2 + 1)(x^2 + 2) \cdots (x^2 + 1009)$$
with subsequent polynomials defined by $p_{k+1}(x) := p_k(x+1) - p_k(x)$ for $k \geq 0$. Find the least n such that
$$p_n(1) = p_n(2) = \cdots = p_n(5000).$$

10. Recall that $\arcsin(t)$ (also known as $\sin^{-1}(t)$) is a function with domain $[-1, 1]$ and range $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Consider the function $f(x) := \arcsin(\sin(x))$ and answer the following questions as a series of four letters (T for True and F for False) in order.

   (a) The function $f(x)$ is well defined for all real numbers x.

   (b) The function $f(x)$ is continuous wherever it is defined.

   (c) The function $f(x)$ is differentiable wherever it is continuous.

Answers to part A

This is the only place that will be seen for grading part A. So carefully and clearly write the answers to each question on the designated line below. Write only the final answers, do not show any intermediate work. Illegible/unclear answers will not be considered. ×

A1. ____________________

A2. ____________________

A3. ____________________

A4. ____________________

A5. ____________________

A6. ____________________

A7. ____________________

A8. ____________________

A9. ____________________

A10. ____________________

Part B. Write complete solutions for these questions from page 6 onwards.

Part B is worth a total of 85 *points (Question 1 is worth* 10 *points and the remaining questions are worth* 15 *points each). Solve these questions in the space provided for each question from page 6. You may solve only part of a question and get partial credit.* **Clearly explain your entire reasoning. No credit will be given without reasoning.**

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1. Answer the following questions

   (a) A natural number k is called **stable** if there exist k *distinct* natural numbers $a_1, \ldots, a_k$, each $a_i > 1$, such that
   $$\frac{1}{a_1} + \cdots + \frac{1}{a_k} = 1.$$
   Show that if k is stable then $k+1$ is also stable. Using this or otherwise, find all stable numbers. [5 marks]

   (b) Let f be a differentiable function defined on a subset A of the real numbers. Define
   $$f^*(y) := \max_{x \in A} \{yx - f(x)\},$$
   whenever the above maximum is finite.
   For the function $f(x) = -\ln(x)$, determine the set of points for which f^* is defined and find an expression for $f^*(y)$ involving only y and constants. [5 marks]

2. Answer the following questions

   (a) Find all real solutions of the equation [6 marks]
   $$\left(x^2 - 2x\right)^{x^2 + x - 6} = 1.$$
   Explain why your solutions are the only solutions.

   (b) The following expression is a rational number. Find its value. [9 marks]
   $$\sqrt[3]{6\sqrt{3} + 10} - \sqrt[3]{6\sqrt{3} - 10}.$$

3. Let f be a function on nonnegative integers defined as follows
   $$f(2n) = f(f(n)) \quad \text{and} \quad f(2n+1) = f(2n) + 1.$$

   (a) If $f(0) = 0$, find $f(n)$ for every n. [2 marks]

   (b) Show that $f(0)$ cannot equal 1. [4 marks]

   (c) For what nonnegative integers k (if any) can $f(0)$ equal 2^k? [9 marks]

4. Let ABC be an equilateral triangle with side length 2. Point A' is chosen on side BC such that the length of $A'B$ is $k < 1$. Likewise points B' and C' are chosen on sides CA and AB with $CB' = AC' = k$. Line segments are drawn from points A', B', C' to their corresponding opposite vertices. The intersections of these line segments form a triangle, labeled PQR. Show that PQR is an equilateral triangle with side length $\frac{4(1-k)}{\sqrt{k^2-2k+4}}$. **[See the figure on page 12.]**

5. An alien script has n letters $b_1, \ldots, b_n$. For some $k < n/2$ assume that all words formed by any of the k letters (written left to right) are meaningful. These words are called *k-words*. A k-word is considered **sacred** if:

   i) no letter appears twice and,

   ii) if a letter b_i appears in the word then the letters b_{i-1} and b_{i+1} do not appear. (Here $b_{n+1} = b_1$ and $b_0 = b_n$.)

   For example, if $n = 7$ and $k = 3$ then $b_1b_3b_6, b_3b_1b_6, b_2b_4b_6$ are sacred 3-words. On the other hand $b_1b_7b_4, b_2b_2b_6$ are not sacred. What is the total number of sacred k-words? Use your formula to find the answer for $n = 10$ and $k = 4$.

6. Imagine the unit square in the plane to be a carrom board. Assume the *striker* is just a point, moving with no friction (so it goes forever), and that when it hits an edge, the angle of reflection is equal to the angle of incidence, as in real life. If the striker ever hits a corner it falls into the pocket and disappears. The trajectory of the striker is completely determined by its starting point (x, y) and its initial velocity $\overrightarrow{(p, q)}$.

   If the striker eventually returns to its initial state (i.e. initial position *and* initial velocity), we define its *bounce number* to be the number of edges it hits before returning to its initial state for the first time.

   For example, the trajectory with initial state $[(.5, .5); \overrightarrow{(1, 0)}]$ has bounce number 2 and it returns to its initial state for the first time in 2 time units. And the trajectory with initial state $[(.25, .75); \overrightarrow{(1, 1)}]$ has bounce number 4. **[See the figures on page 16.]**

   (a) Suppose the striker has initial state $[(.5, .5); \overrightarrow{(p, q)}]$. If $p > q \geq 0$ then what is its velocity after it hits an edge for the first time? What if $q > p \geq 0$? [2 marks]

   (b) Draw a trajectory with bounce number 5 or justify why it is impossible. [3 marks]

   (c) Consider the trajectory with initial state $[(x, y); \overrightarrow{(p, 0)}]$ where p is a positive integer. In how much time will the striker first return to its initial state? [2 marks]

   (d) What is the bounce number for the initial state $[(x, y); \overrightarrow{(p, q)}]$ where p, q are relatively prime positive integers, assuming the striker never hits a corner? [8 marks]

Write answers to part B from the next page.

Answers to part B

If you need extra space for any problem, continue on one of the colored blank pages at the end and write a note to that effect.

1. Answer the following questions

   (a) A natural number k is called **stable** if there exist k *distinct* natural numbers $a_1, \dots, a_k$, each $a_i > 1$, such that

   $$\frac{1}{a_1} + \cdots + \frac{1}{a_k} = 1.$$

   Show that if k is stable then $k+1$ is also stable. Using this or otherwise, find all stable numbers. [5 marks]

(b) Let f be a differentiable function defined on a subset A of the real numbers. Define

$$f^*(y) := \max_{x \in A} \{yx - f(x)\},$$

whenever the above maximum is finite.

For the function $f(x) = -\ln(x)$, determine the set of points for which f^* is defined and find an expression for $f^*(y)$ involving only y and constants. [5 marks]

2. Answer the following questions

   (a) Find all real solutions of the equation [6 marks]

   $$\left(x^2 - 2x\right)^{x^2 + x - 6} = 1.$$

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$$\sqrt[3]{6\sqrt{3}+10}-\sqrt[3]{6\sqrt{3}-10}.$$

3. Let f be a function on nonnegative integers defined as follows

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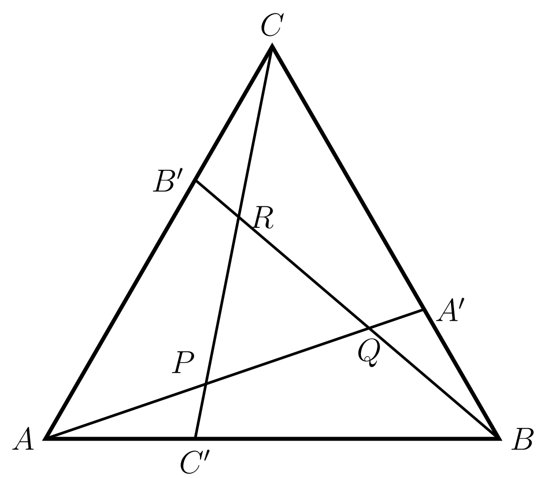

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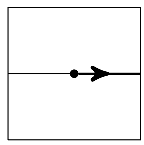 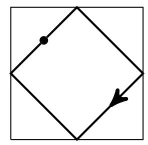

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