Problems in Combinatorics

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✻ Problem 1

(IMO, 1989) A permutation $x_1x_2\ldots x_{2n}$ of the set $\{1,2,\ldots,2n\}$, where $n\in\mathbf{N}$, is said to have property P if $|x_i-x_{i+1}|=n$ for at least one i in $\{1,2,\ldots,2n-1\}$. Show that, for each n, there are more permutations with property P than without.

Solution: The case when $n=1$ is trivial. Assume that $n\geq 2$. Let A (resp., B) be the set of permutations of $S=\{1,2,\ldots,2n\}$ without property P (resp., with P). To show that $|B|>|A|$, by (IP) and (BP), it suffices to establish a mapping $f:A\to B$ which is injective but not surjective.

For convenience, any number in the pair $\{k,n+k\}(k=1,2,\ldots,n)$ is called the partner of the other. If k and $n+k$ are adjacent in a permutation, the pair $\{k,n+k\}$ is called an adjacent pair of partners.

Let $\alpha=x_1x_2\ldots x_{2n}$ be an element in A. Since α does not have property P, the partner of x_1 is x_r where $3\leq r\leq 2n$. Now we put

$$f(\alpha)=x_2x_3\ldots x_{r-1}\underline{x_1x_r}x_{r+1}\ldots x_{2n}$$

by taking x_1 away and placing it just in front of its partner x_r. In $f(\alpha)$, it is clear that $\{x_1,x_r\}$ is the only adjacent pair of partners. Obviously, $f(\alpha)\in B$ and f defines a mapping from A to B. We now claim that f is injective. Let

$$\alpha=x_1x_2\ldots x_{2n}$$
$$\beta=y_1y_2\ldots y_{2n}$$

be elements of A in which x_1 's partner is x_r and y_1 's partner is y_s, where $3\leq r,s\leq 2n$. Suppose $f(\alpha)=f(\beta)$; i.e.,

$$x_2x_3\ldots x_{r-1}\underline{x_1x_r}\ldots x_{2n}=y_2y_3\ldots y_{s-1}\underline{y_1y_s}\ldots y_{2n}.$$

Since $\{x_1,x_r\}$ (resp., $\{y_1,y_s\}$) is the only adjacent pair of partners in $f(\alpha)$ (resp., $f(\beta)$), we must have $r=s, x_1=y_1$ and $x_r=y_s$. These, in turn, imply that $x_i=y_i$ for all $i=1,2,\ldots,2n$ and so $\alpha=\beta$, showing that f is injective.

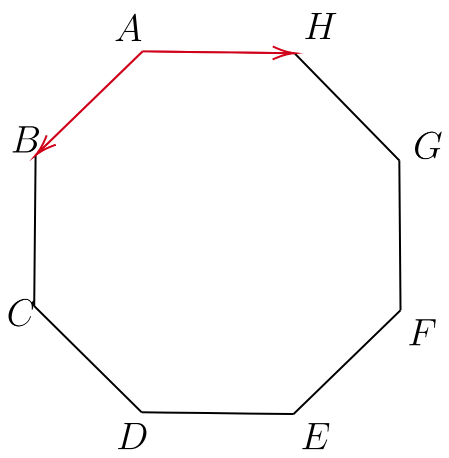
Finally, we note that $f(A)$ consists of all permutations of S having exactly one adjacent pair of partners while there are permutations of S in B which contain more than one adjacent pair of partners. Thus we have $f(A)\subset B$, showing that f is not surjective. The proof is thus complete. ■

✻ Problem 2

Let A and E be opposite vertices of a regular octagon (Figure 5.3). A frog starts jumping at vertex A. From any vertex of the octagon except E, it may jump to either of the two adjacent vertices. When it reaches vertex E, the frog stops and stays there. Let u_n be the number of distinct paths of exactly n jumps ending at E. Prove that $u_{2n-1} = 0$,

$$u_{2n} = \frac{1}{\sqrt{2}}\left(x^{n-1} - y^{n-1}\right), \quad n = 1, 2, 3, \ldots,$$

where $x = 2 + \sqrt{2}$ and $y = 2 - \sqrt{2}$. (Note that a path of n jumps is a sequence of vertices $(P_0, \ldots, P_n)$ such that
(i) $P_0 = A, P_n = E$;
(ii) for every $i, 0 \leq i \leq n-1, P_i$ is distinct from E; and
(iii) for every $i, 0 \leq i \leq n-1, P_i$ and P_{i+1} are adjacent.)



Solution: Let $a_n, b_n, c_n, d_n, e_n, f_n, g_n$, and h_n be the number of distinct paths of exactly n jumps ending at A, B, C, D, E, F, G, and H, respectively. Then $e_n = u_n$. By symmetry, we know that $b_n = h_n$, $c_n = g_n$, and $d_n = f_n$. It is also not difficult to see that

$$\begin{aligned}
e_n &= d_{n-1} + f_{n-1} = 2d_{n-1} \\
d_n &= c_{n-1} \\
a_n &= 2b_{n-1} \\
c_n &= b_{n-1} + d_{n-1} \\
b_n &= a_{n-1} + c_{n-1}
\end{aligned}$$

From the first two equations, we obtain $d_{n-1} = \frac{1}{2}e_n$ and $c_{n-1} = d_n = \frac{1}{2}e_{n+1}$. Substituting these two relations into the fourth equation gives $b_{n-1} = \frac{1}{2}\left(e_{n+2} - e_n\right)$.

From the third equation, we obtain $a_n = e_{n+2} - e_n$. Then the fifth equation reads

$$e_{n+3} - 4e_{n+1} + 2e_{n-1} = 0.$$

Thus we may consider the characteristic equation

$$x^4 - 4x^2 + 2 = 0$$

The four roots are $\pm\sqrt{2 \pm \sqrt{2}}$.
$u_n = e_n = as^n + b(-s)^n + ct^n + d(-t)^n$ where $s = \sqrt{x} = \sqrt{2+\sqrt{2}}$ and $t = \sqrt{y} = \sqrt{2-\sqrt{2}}$. Note that $e_1 = e_2 = e_3 = 0$ and $e_4 = 2$. Hence

$$\begin{aligned} (a-b)s + (c-d)t &= 0 \\ (a+b)s^2 + (c+d)t^2 &= 0 \\ (a-b)s^3 + (c-d)t^3 &= 0 \\ (a+b)s^4 + (c+d)t^4 &= 2 \end{aligned}$$

From the first and third equations, we obtain $a = b$ and $c = d$. This implies that $u_{2n-1} = 0$. Solving the second and fourth equations gives

$$a = b = \frac{t^2}{4\sqrt{2}} \quad \text{and} \quad c = d = -\frac{s^2}{4\sqrt{2}}.$$

Consequently, $u_{2n} = 2as^{2n} + 2ct^{2n} = \dfrac{s^2t^2}{2\sqrt{2}}(s^{2n-2} - t^{2n-2}) = \dfrac{1}{\sqrt{2}}(x^{n-1} - y^{n-1})$. ■

✻ Problem 3

For a set A, let $s(A)$ denote the sum of the elements of A. (If $A = \emptyset$, then $|A| = s(A) = 0$.) Let

$$S = \{1, 2, \ldots, 1999\}.$$

For $r = 0, 1, 2, \ldots, 6$, define

$$T_r = \{T \mid T \subseteq S, s(T) \equiv r \pmod 7\}$$

For each r, find the number of elements in T_r.

Solution: If an integer $i, 1 \leq i \leq 1999$, is in T, it contributes i in the sum $s(T)$; otherwise, it contributes 0 . Hence for each number i, we associate it with the generating function $x^0 + x^i = 1 + x^i$. Consider the polynomial

$$f(x) = (1+x)\left(1+x^2\right)\cdots\left(1+x^{1999}\right) = \sum_n c_n x^n$$

Then there is a bijection between each subset $T=\{a_1,a_2,\ldots,a_m\}$ of S and each term $x^{a_1}x^{a_2}\cdots x^{a_m}=x^{a_1+a_2+\cdots+a_m}$. Hence

$$|T_r|=\sum_k\left[x^{7k+r}\right]f(x)=\sum_k c_{7k+r}.$$

Let $\xi=e^{\frac{2\pi}{7}i}$, where $i^2=-1$, be a 7^{th} root of unity. Then $\xi^7=1$ and $\xi\neq 1$, so ξ is a root of

$$\frac{x^7-1}{x-1}=1+x+x^2+\cdots x^6$$

That is, $1+\xi+\xi^2+\cdots+\xi^6=0$. For r divisible by 7 , we have $\sum_{k=1}^{6}\xi^{kr}=\sum_{k=1}^{6}1=6$. For r not divisible by 7 ,

$$\{1,2,\ldots,6\}\equiv\{r\cdot 1,r\cdot 2,\cdots,r\cdot 6\}\pmod 7$$

(In other words, a complete set of residue classes modulo 7 remains invariant by multiplying r by each number in the set.) Thus,

$$\sum_{k=1}^{6}\xi^{kr}=\begin{cases}6, & r \text{ is divisible by } 7,\\ -1, & r \text{ is not divisible by } 7.\end{cases}$$

Hence,

$$\sum_{i=0}^{6}f\left(\xi^i\right)=\sum_{i=0}^{6}\sum_n c_n\xi^{ni}=\sum_n c_n\sum_{i=0}^{6}\xi^{ni}=\sum_{7|n}7c_n=7\left|T_0\right|$$

In exactly the same way, we can show that

$$|T_r|=\frac{1}{7}\sum_{i=0}^{6}\xi^{-ri}f\left(\xi^i\right)=\frac{1}{7}\left(2^{1999}+\sum_{i=1}^{6}\xi^{-ri}f\left(\xi^i\right)\right)$$

since $f\left(\xi^0\right)=f(1)=2^{1999}$. Note also that $\xi,\xi^2,\ldots,\xi^7=1$ are the roots of $g(x)=x^7-1$, that is,

$$g(x)=x^7-1=(x-\xi)\left(x-\xi^2\right)\cdots\left(x-\xi^7\right)$$

It follows that

$$g(-1)=-2=(-1-\xi)\left(-1-\xi^2\right)\left(-1-\xi^3\right)\cdots\left(-1-\xi^7\right)$$

implying that

$$(1+\xi)\left(1+\xi^2\right)\cdots\left(1+\xi^7\right)=2$$

Consequently, because $1999 = 7 \cdot 285 + 4$, we have

$$\begin{aligned} f(\xi) &= (1+\xi)\left(1+\xi^2\right)\cdots\left(1+\xi^{1999}\right) \\ &= \left[(1+\xi)\left(1+\xi^2\right)\cdots\left(1+\xi^7\right)\right]^{285}(1+\xi)\left(1+\xi^2\right)\left(1+\xi^3\right)\left(1+\xi^4\right) \\ &= 2^{285}\cdot\left[(1+\xi)\left(1+\xi^2\right)\left(1+\xi^4\right)\right]\left(1+\xi^3\right) \\ &= 2^{285}\cdot\left(1+\xi+\xi^2+\cdots+\xi^7\right)\left(1+\xi^3\right) \\ &= 2^{285}\left(1+\xi^3\right) \end{aligned}$$

In general, we have $f\left(\xi^i\right) = 2^{285}\left(1+\xi^{3i}\right)$ for $1 \le i \le 6$. It follows that

$$\begin{aligned} |T_r| &= \frac{1}{7}\left(2^{1999} + 2^{285}\sum_{i=1}^{6}\xi^{-ri}\left(1+\xi^{3i}\right)\right) \\ &= \frac{1}{7}\left(2^{1999} + 2^{285}\sum_{i=1}^{6}\left[\xi^{-ri}+\xi^{(3-r)2}\right]\right) \end{aligned}$$

By equation (**), we conclude that

$$\sum_{i=1}^{6}\left[\xi^{-ri}+\xi^{(3-r)i}\right] = \begin{cases} 6-1=5, & r \equiv 0 \text{ or } 3 \pmod 7 \\ -1-1=-2, & \text{otherwise.} \end{cases}$$

Therefore, the answer to the problem is

$$|T_r| = \begin{cases} \frac{2^{1999}+5\cdot 2^{285}}{7} & r = 0 \text{ or } 3, \\ \frac{2^{1999}-2^{286}}{7} & r = 1,2,4,5,6. \end{cases}$$