Limit, Continuity and Differentiability of Functions
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In this chapter we shall study limit and continuity of real valued functions defined on certain sets.

2.1 Limit of a Function

Suppose f is a real valued function defined on a subset D of $\mathbb{R}$. We are going to define *limit* of $f(x)$ as $x \in D$ *approaches* a point a which is not necessarily in D.

First we have to be clear about what we mean by the statement "$x \in D$ *approaches* a point a".

2.1.1 Limit point of a set $D \subseteq \mathbb{R}$

Definition 2.1 Let $D \subseteq \mathbb{R}$ and $a \in \mathbb{R}$. Then a is said to be a **limit point** of D if for any $\delta > 0$, the interval $(a - \delta,\, a + \delta)$ contains atleast one point from D other than possibly a, i.e.,

$$D \cap \{x \in R : 0 < |x - a| < \delta\} \neq \varnothing.$$

□

Example 2.1 The statements in the following can be easily verified:

(i) Every point in an interval is its limit point.

(ii) If I is an open interval of finite length, then both the end points of I are limit points of I.

(iii) The set of all limit points of an interval I of finite length consists of points from I together with its endpoints.

(iv) If $D = \{x \in \mathbb{R} : 0 < |x| < 1\}$, then every point in the interval $[-1, 1]$ is a limit point of D.

(v) If $D = (0, 1) \cup \{2\}$, then 2 is not a limit point of D. The set of all limit points of D is the closed interval $[0, 1]$.

(vi) If $D = \{\frac{1}{n} : n \in \mathbb{N}\}$, then 0 is the only limit point of D.

(vii) If $D = \{n/(n+1) : n \in \mathbb{N}\}$, then 1 is the only limit point of D. □

For the later use, we introduce the following definition.

Definition 2.2 (i) For $a \in \mathbb{R}$, an open interval of the form $(a - \delta,\, a + \delta)$ for some $\delta > 0$ is called a **neighbourhood** of a; it is also called a **δ-neighbourhood** of a.

(ii) By a **deleted neighbourhood** of a point $a \in \mathbb{R}$ we mean a set of the form $D_\delta := \{x \in \mathbb{R} : 0 < |x - a| < \delta\}$ for some $\delta > 0$, i.e., the set $(a - \delta, a + \delta) \setminus \{a\}$. □

With the terminologies in the above definition, we can state the followng:

- A point $a \in \mathbb{R}$ is a limit point of $D \subseteq \mathbb{R}$ if and only if every deleted neighbourhood of a contains at least one point of D.

  In particular, if D contains either a deleted neighbourhood of a or if D contains an open interval with one of its end points is a, then a is a limit point of D.

Now we give a characterization of limit points in terms of convergence of sequences.

Theorem 2.1 *A point $a \in \mathbb{R}$ is a limit point of $D \subseteq \mathbb{R}$ if and only if there exists a sequence (a_n) in $D \setminus \{a\}$ such that $a_n \to a$ as $n \to \infty$.*

Proof. Suppose $a \in \mathbb{R}$ is a limit point of D. Then for each $n \in \mathbb{N}$, there exists $a_n \in D \setminus \{a\}$ such that $a_n \in (a - 1/n, a + 1/n)$. Note that that $a_n \to a$.

Conversely, suppose that there exists a sequence (a_n) in $D \setminus \{a\}$ such that $a_n \to a$. Hence, for every $\delta > 0$, there exists $N \in \mathbb{N}$ such that $a_n \in (a - \delta, a + \delta)$ for all $n \geq N$. In particular, for $n \geq N$, $a_n \in (a - \delta, a + \delta) \cap (D \setminus \{a\})$. ∎

Exercise 2.1 Prove that a point $a \in \mathbb{R}$ is a limit point of $D \subset \mathbb{R}$ if and only if there exists a sequence (a_n) in D such that (a_n) is not eventually constant and $a_n \to a$ as $n \to \infty$. [Recall that a sequence (a_n) is said to be eventually constant if there exists $k \in \mathbb{N}$ such that $a_n = a_k$ for all $n \geq k$.] ◀

2.1.2 Limit of a function $f(x)$ as x approaches a

Definition 2.3 Let f be a real valued function defined on a set $D \subseteq \mathbb{R}$, and let $a \in \mathbb{R}$ be a limit point of D. We say that $b \in \mathbb{R}$ is a **limit of $f(x)$ as x approaches** a if for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(x) - b| < \varepsilon \quad \text{whenever} \quad x \in D,\ 0 < |x - a| < \delta, \tag{$*$}$$

and in that case we write

$$\lim_{x \to a} f(x) = b$$

or

$$f(x) \to b \quad \text{as} \quad x \to a.$$

□

The relations in (∗) in the above examples can also be written as

$$x \in D, \quad 0 < |x - a| < \delta \quad \Longrightarrow \quad |f(x) - b| < \varepsilon.$$

Exercise 2.2 Thus, $\lim_{x \to a} f(x) = b$ if and only if for every open interval I_b containing b there exists an open interval I_a containing a such that

$$x \in I_a \cap (D \setminus \{a\}) \quad \Longrightarrow \quad f(x) \in I_b.$$

◀

CONVENTION: In the following, whenever we talk about limit of a function f as x approaches $a \in \mathbb{R}$, we assume that f is defined on a set $D \subseteq \mathbb{R}$ and a is a limit point of D.

Also, when we talk about $f(x)$, we assume that x belongs to the domain of f. For example, if we say that "$f(x)$ has certain property P for every x in an interval I", what we mean actualy is that "$f(x)$ has the property P for all $x \in I \cap D$, where D is he domain of f".

Exercise 2.3 Show that, a function cannot have more than one limits. ◀

Example 2.2 Let D be an interval and a is either in D or a is an end point of D.

(i) Let $f(x) = x$. Since

$$|f(x) - a| = |x - a| \quad \forall\, x \in D,$$

it follows that for any $\varepsilon > 0$, $|f(x) - a| < \varepsilon$ whenever $0 < |x - a| < \delta := \varepsilon$. Hence, $\lim_{x \to a} f(x) = a$.

(ii) Let $f(x) = x^2$ and $\varepsilon > 0$ be given. We show that $\lim_{x \to a} f(x) = a^2$. Note that

$$|f(x) - a^2| = (|x| + |a|)|x - a| \quad \forall\, x \in D,\, x \neq a.$$

Since $|x| \leq |x - a| + |a| \leq 1 + |a|$ whenever $|x - a| < 1$, we have

$$|f(x) - a^2| = (1 + 2|a|)|x - a| \quad \forall\, x \in D,\, 0 < |x - a| \leq 1.$$

Therefore,

$$x \in D,\, 0 < |x - a| \leq 1,\, (1 + 2|a|)|x - a| < \varepsilon \quad \Longrightarrow \quad |f(x) - a^2| < \varepsilon.$$

Thus,

$$x \in D,\, 0 < |x - a| < \delta := \min\{1,\, \varepsilon/(1 + 2|a|)\} \quad \Longrightarrow \quad |f(x) - a^2| < \varepsilon.$$

Hence, $\lim_{x \to a} f(x) = a^2$. □

More examples will be considered in Section 2.1.4 after proving some properties of the limit. Before that let us ask the following question.

Question: Suppose f is a real valued function defined on an interval I and $a \in I$. What do we mean by the statement that "$\lim_{x \to a} f(x)$ does not exist"?

It means the following: For any $b \in \mathbb{R}$, there exists $\varepsilon > 0$ such that for any $\delta > 0$, there is atleast one $x_\delta \in (a - \delta, a + \delta)$ such that $f(x_\delta) \notin (b - \varepsilon, b + \varepsilon)$.

We illustrate this by a simple example.

Example 2.3 Let $f : [-1, 1] \to \mathbb{R}$ be defined by $f(x) = \begin{cases} 0, & -1 \le x \le 0, \\ 1, & 0 < x \le 1. \end{cases}$ We show that $\lim_{x \to 0} f(x)$ does not exist. For this let $b \in \mathbb{R}$. Let us consider the following cases:

Case (i): $b = 0$. In this case, if $0 < \varepsilon < 1$, then $(b - \varepsilon, b + \varepsilon)$ does not contain 1 so that $f(x) \notin (b - \varepsilon, b + \varepsilon)$ for any $x > 0$.

Case (ii): $b = 1$. In this case, if $0 < \varepsilon < 1$, then $(b - \varepsilon, b + \varepsilon)$ does not contain 0 so that $f(x) \notin (b - \varepsilon, b + \varepsilon)$ for any $x < 0$.

Case (iii): $b \ne 0$, $b \ne 1$. In this case, if $0 < \varepsilon < \min\{|b|, |b - 1|\}$, then $(b - \varepsilon, b + \varepsilon)$ does not contain 0 and 1 so that $f(x) \notin (b - \varepsilon, b + \varepsilon)$ for any $x \ne 0$.

Thus, b is not a limit of $f(x)$ as x approaches 0. □

Before going further, let us observe a property which would be used in the due course.

Theorem 2.2 *If $\lim_{x \to a} f(x) = b$, then there exists a deleted neighbourhood D_δ of a and $M > 0$ such that $|f(x)| \le M$ for all $x \in D_\delta \cap D$.*

Proof. Suppose $\lim_{x \to a} f(x) = b$. Then there exists a deleted neighbourhood D_δ of a such that $|f(x) - b| < 1$ for all $x \in D \cap D_\delta$. Hence,

$$|f(x)| \le |f(x) - b| + |b| < 1 + |b| \quad \forall\, x \in D \cap D_\delta.$$

Thus, $|f(x)| \le M = 1 + |b|$ for all $x \in D_\delta \cap D$. ∎

2.1.3 Limit of a function in terms of sequences

Let a be a limit point of $D \subseteq \mathbb{R}$ and $f : D \to \mathbb{R}$.

Suppose $\lim_{x \to a} f(x) = b$. Since a is a limit point of D, we know that there exists a sequence (x_n) in $D \setminus \{a\}$ such that $x_n \to a$. Does $f(x_n) \to b$? The answer is "yes". In fact, we have more!

Theorem 2.3 *If $\lim_{x \to a} f(x) = b$, then for every sequence (x_n) in D such that $x_n \to a$, we have $f(x_n) \to b$.*

Proof. Suppose $\lim_{x\to a} f(x) = b$. Let (x_n) be a sequence in D such that $x_n \to a$. Let $\varepsilon > 0$ be given. We have to show that there exists $n_0 \in \mathbb{N}$ such that $|f(x_n) - b| < \varepsilon$ for all $n \geq n_0$.

Since $\lim_{x\to a} f(x) = b$, we know that there exists $\delta > 0$ such that

$$x \in D,\ 0 < |x - a| < \delta \Longrightarrow |f(x) - b| < \varepsilon. \qquad (*)$$

Also, since $x_n \to a$, there exists $n_0 \in \mathbb{N}$ such that $|x_n - a| < \delta$ for all $n \geq n_0$. Hence, from $(*)$, we have $|f(x_n) - b| < \varepsilon$ for all $n \geq n_0$. ∎

The converse of the above theorem is also true.

Theorem 2.4 *If for every sequence (x_n) in D which converges to a, the sequence $(f(x_n))$ converges to b, then $\lim_{x\to a} f(x) = b$.*

Proof. Suppose for every sequence (x_n) in D which converges to a, the sequence $(f(x_n))$ converges to b. Assume for a moment that f does not have the limit b as x approaches a. Then, by the definition of the limit, there exists $\varepsilon_0 > 0$ such that for every $\delta > 0$, there exists at least one $x_\delta \in D$ such that

$$0 < |x_\delta - a| < \delta \quad \text{and} \quad |f(x_\delta) - b| > \varepsilon_0.$$

In particular, for every $n \in \mathbb{N}$, there exists $x_n \in D$ such that

$$0 < |x_n - a| < \frac{1}{n} \quad \text{and} \quad |f(x_n) - b| > \varepsilon_0.$$

Thus, $x_n \to a$ but $f(x_n) \not\to b$. This is a contradiction to our hypothesis. ∎

Remark 2.1 Here are some implications of the first part of Theorem 2.3. Suppose (x_n) is a a sequence in $D \setminus \{a\}$ such that $x_n \to a$.

1. If $(f(x_n))$ does not converge, then $\lim_{x\to a} f(x)$ does not exist.
2. If $(f(x_n))$ does not converge to a given $b \in \mathbb{R}$, then either $\lim_{x\to a} f(x)$ does not exist or $\lim_{x\to a} f(x)$ exists but $\lim_{x\to a} f(x) \neq b$.
3. If (y_n) is another sequence in $D \setminus \{a\}$ which converges to a and the sequences $(f(x_n))$ and $(f(y_n))$ converge to different points, then $\lim_{x\to a} f(x)$ does not exist.

If we are able to show the convergence of $(f(x_n))$ to some b for any arbitrary (*not for a specific*) sequence (x_n) in $D \setminus \{a\}$ which converges to a, then by second part of Theorem 2.3, we can assert that $\lim_{x\to a} f(x) = b$. ♦

Example 2.4 Consider the function f in Example 2.3, i.e., $f : [-1, 1] \to \mathbb{R}$ is defined by $f(x) = \begin{cases} 0, & -1 \le x \le 0, \\ 1, & 0 < x \le 1. \end{cases}$

Suppose (x_n) is a sequence of negative numbers and (y_n) is a sequence of positive numbers such that both of them converge to 0. Then we have $f(x_n) = 0$ and $f(y_n) = 1$ for all $n \in \mathbb{N}$. Hence, $\lim_{n\to\infty} f(x_n)$ and $\lim_{n\to\infty} f(y_n)$ exist, but they are different. Hence $\lim_{x\to 0} f(x)$ does not exist. □

2.1.4 Some properties

The following two theorems can be proved using Theorems 2.3 and 2.4, and the results on convergence of sequences of real numbers.

Theorem 2.5 *We have the following.*

(i) *If* $\lim_{x\to a} f(x) = b$ *and* $\lim_{x\to a} g(x) = c$, *then*

$$\lim_{x\to a}[f(x) + g(x)] = b + c, \qquad \lim_{x\to a} f(x)g(x) = bc.$$

(ii) *If* $\lim_{x\to a} f(x) = b$ *and* $b \ne 0$, *then* $f(x) \ne 0$ *in a deleted neighbourhood of* a *and*

$$\lim_{x\to a} \frac{1}{f(x)} = \frac{1}{b}.$$

Theorem 2.6 (Sandwich theorem) *If* f *and* g *have the same limit* b *as* x *approaches* a, *and if* h *is a function such that* $f(x) \le h(x) \le g(x)$ *for all* x *in a deleted neighbourhood of* a, *then* $\lim_{x\to a} h(x) = b$.

The following two corollaries are immediate from Theorem 2.5.

Corollary 2.7 *If* $\lim_{x\to a} f(x) = b$, $\lim_{x\to a} g(x) = c$, *and* $c \ne 0$, *then* g *is nonzero in a deleted neighbourhood of* c *and*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{b}{c}.$$

Corollary 2.8 *If* $\lim_{x\to a} f(x) = b$, $\lim_{x\to a} g(x) = c$ *and* $f(x) \ge g(x)$ *for all* x *in a deleted neighbourhood of* a, *then* $b \ge c$.

Exercise 2.4 Write detailed proof of Theorem 2.5, Theorem 2.6 and Corollary 2.7 and Corollary 2.8. ◄

Theorem 2.9 *Suppose* $\lim_{x\to a} f(x) = b$ *and* $\lim_{y\to b} g(y) = c$. *If* D_1 *and* D_2 *are the domains of* f *and* g *respectively, and if* $f(x) \in D_2 \setminus \{b\}$ *for every* $x \in D_1 \setminus \{a\}$, *then* $\lim_{x\to a} g(f(x)) = c$.

Proof. By Theorem 2.6, it is enough to prove that for any sequence (x_n) in $D_1\setminus\{a\}$ such that $x_n \to a$, we have $g(f(x_n)) \to c$. So, let (x_n) be in $D_1\setminus\{0\}$ such that $x_n \to a$. Since $\lim_{x\to a} f(x) = b$, by Theorem 2.5, $f(x_n) \to b$. Let $y_n = f(x_n)$, $n \in \mathbb{N}$. By the assumption, $y_n \in D_2 \setminus \{b\}$ for all $n \in \mathbb{N}$. Since $\lim_{y\to b} g(y) = c$ and $y_n \to b$, again by Theorem 2.5, $g(y_n) \to c$. Thus we obtained $g(f(x_n) \to c$, which completes the proof. ∎

Alternate proof using $\varepsilon - \delta$ arguments. Let $\varepsilon > 0$ be given. Then there exists $\delta_1 > 0$ such that

$$0 < |y - b| < \delta_1 \Longrightarrow |g(y) - c| < \varepsilon.$$

Also, let $\delta_2 > 0$ be such that

$$0 < |x - a| < \delta_2 \Longrightarrow |f(x) - b| < \delta_1.$$

Hence, along with the given condition that $f(x) \in D_2 \setminus \{b\}$ for every $x \in D_1 \setminus \{a\}$,

$$0 < |x - a| < \delta_2 \Longrightarrow 0 < |f(x) - b| < \delta_1 \Longrightarrow |g(f(x)) - c| < \varepsilon.$$

This completes the proof. ∎

Exercise 2.5 Suppose φ is a function defined in a neighbourhood of a point x_0 such that $\lim_{x\to x_0} \varphi(x) = x_0$. If f is also a function defined in a neighbourhood of x_0 and $\lim_{x\to x_0} f(x)$ exists, then prove that $\lim_{x\to x_0} f(\varphi(x))$ exists and

$$\lim_{x\to x_0} f(\varphi(x)) = \lim_{x\to x_0} f(x).$$

◀

Example 2.5 If $f(x)$ is a polynomial, say $f(x) = a_0 + a_1x + \ldots + a_kx^k$, then for any $a \in \mathbb{R}$,

$$\lim_{x\to a} f(x) = f(a).$$

We obtain this by using Theorem 2.5. Let us show the same by using the definition, i.e., using $\varepsilon - \delta$ arguments: Let $b = f(a)$ and let $\varepsilon > 0$ be given. We have to find $\delta > 0$ such that $|x - a| < \delta \Longrightarrow |f(x) - b| < \varepsilon$. Note that

$$f(x) - f(a) = a_1(x - a) + a_2(x^2 - a^2) + \ldots + a_k(x^k - a^k),$$

where

$$x^n - a^n = (x - a)[x^{n-1} + x^{n-2}a + \ldots + xa^{n-2} + a^{n-1}].$$

Now, suppose $|x - a| < 1$. Then we have $|x| < 1 + |a|$ so that

$$|x^{n-j}a^{j-1}| < (1 + |a|)^{n-1}$$

and hence,

$$|x^n - a^n| < |x - a|n(1 + |a|)^{n-1}.$$

Thus, $|x - a| < 1$ implies

$$|f(x) - f(a)| \leq |x - a|\Big(|a_1| + |a_2|2(1 + |a|) + \ldots + |a_k|k(1 + |a|)^{k-1}\Big),$$

Therefore, taking $\alpha := |a_1| + |a_2|2(1 + |a|) + \ldots + |a_k|k(1 + |a|)^{k-1}$, we have

$$|f(x) - f(a)| < \varepsilon \quad \text{whenever} \quad |x - a| < \delta := \min\{1, \varepsilon/\alpha\}.$$

□

Example 2.6 Let $D = \mathbb{R} \setminus \{2\}$ and $f(x) = \frac{x^2-4}{x-2}$. Then $\lim_{x\to 2} f(x) = 4$.

Note that, for $x \neq 2$,

$$f(x) = \frac{(x + 2)(x - 2)}{x - 2} = (x + 2).$$

Hence, for $\varepsilon > 0$, $|f(x) - 4| < \varepsilon$ whenever $|x - 2| < \delta := \varepsilon$. □

Example 2.7 Let $D = \mathbb{R} \setminus \{0\}$ and $f(x) = \frac{1}{x}$. Then $\lim_{x\to 0} f(x)$ does not exist. To see this consider the sequence (x_n) with $x_n = 1/n$ for $n \in \mathbb{N}$. Then we have $x_n \to 0$ but $\{f(x_n)\}$ diverges to infinity. Therefore, by Theorem 2.3, $\lim_{x\to 0} f(x)$ does not exist.

Alternatively, for any $b \in \mathbb{R}$,

$$|f(x) - b| \geq |f(x)| - |b| > 1 \quad \text{whenever} \quad |f(x)| > 1 + |b|.$$

But,

$$|f(x)| > 1 + |b| \iff |x| < \frac{1}{1 + |b|}.$$

Thus, for any $b \in \mathbb{R}$,

$$|f(x) - b| > 1 \quad \text{whenever} \quad |x| < \frac{1}{1 + |b|}.$$

Thus, we have proved that it is not possible to find a $\delta > 0$ such that $|f(x) - b| < 1$ for all x with $|x| < \delta$. □

Example 2.8 We show that (i) $\lim_{x\to 0} \sin(x) = 0$ and (ii) $\lim_{x\to 0} \cos(x) = 1$.

From the graph of the function $\sin x$, it is clear that

$$-\frac{\pi}{2} < x < 0 \Longrightarrow 0 < |\sin x| < |x|.$$

Hence, from Theorem 2.6, we have $\lim_{x\to 0} |\sin x| = 0$. Thus, $\lim_{x\to 0} \sin(x) = 0$.

Also, since $\cos x = 1 - 2\sin^2(x/2)$ and $\lim_{x\to 0} \sin(x/2) = 0$, Theorem 2.5(i) implies $\lim_{x\to 0} \cos x = 1$. □

Example 2.9 We show that $\lim_{x\to 0} \frac{\sin x}{x} = 1$.

It can be seen, using the graph of $\sin x$ that

$$0 < x < \frac{\pi}{2} \Longrightarrow \sin x < x < \tan x.$$

Hence,

$$0 < x < \frac{\pi}{2} \Longrightarrow \cos x < \frac{\sin x}{x} < 1.$$

Since $\frac{\sin(-x)}{-x} = \frac{\sin x}{x}$ and $\cos(-x) = \cos x$, it follows that

$$0 < |x| < \frac{\pi}{2} \Longrightarrow \cos x < \frac{\sin x}{x} < 1.$$

Therefore, by Theorem 2.5(iv) and Example 2.8(ii), we have $\lim_{x\to 0} \frac{\sin x}{x} = 1$. □

Remark 2.2 In the above two examples we have used some properties of the functions $\sin x$, $\cos x$ and $\tan x$, though we have not defined these functions formally. We shall define these functions formally in the due course. ♦

Exercise 2.6 Let $f : \mathbb{R} \to \mathbb{R}$ be such that $f(x + y) = f(x) + f(y)$. Suppose $\lim_{x\to 0} f(x)$ exists. Prove that $\lim_{x\to 0} f(x) = 0$ and $\lim_{x\to c} f(x) = f(c)$ for every $c \in \mathbb{R}$.

Hint: Use the facts that $f(2x) = 2f(x)$, Theorem 2.9 and $f(x) - f(c) = f(x - c)$. ◄

Exercise 2.7 Suppose φ is a function defined in a neighbourhood I_0 of a point x_0 such that

$$x \in I_0,\ |x - x_0| < r \Longrightarrow |\varphi(x) - x_0| < r \qquad \forall\, r > 0.$$

If f is also a function defined in a neighbourhood of x_0 and $\lim_{x\to x_0} f(x)$ exists, then prove that $\lim_{x\to x_0} f(\varphi(x))$ exists and $\lim_{x\to x_0} f(\varphi(x)) = \lim_{x\to x_0} f(x)$. ◄

2.1.5 Left limit and right limit

Definition 2.4 Let f be a real valued function defined on a set $D \subseteq \mathbb{R}$, and let $a \in \mathbb{R}$ be a limit point of D.

(i) We say that $f(x)$ **has the left limit** $b \in \mathbb{R}$ **as** x **approaches** a if for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(x) - b| < \varepsilon \quad \text{whenever} \quad x \in D,\ a - \delta < x < a,$$

and in that case we write

$$\lim_{x\to a^-} f(x) = b$$

or

$$f(x) \to b \quad \text{as} \quad x \to a^-.$$

(ii) We say that $f(x)$ **has the right limit** $b \in \mathbb{R}$ **as** x **approaches** a if for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(x) - b| < \varepsilon \quad \text{whenever} \quad x \in D,\ a < x < a + \delta,$$

and in that case we write

$$\lim_{x \to a^+} f(x) = b$$

or

$$f(x) \to b \quad \text{as} \quad x \to a^+.$$

□

We have the following characterizations in terms of sequences (*Verify*):

1. $\lim_{x \to a^-} f(x) = b$ if and only if for every sequence (x_n) in $D \setminus \{a\}$,

$$x_n < a \quad \forall\, n \in \mathbb{N}, \quad x_n \to a \quad \Longrightarrow \quad f(x_n) \to b.$$

2. $\lim_{x \to a^+} f(x) = b$ if and only if for every sequence (x_n) in $D \setminus \{a\}$,

$$x_n > a \quad \forall\, n \in \mathbb{N}, \quad x_n \to a \quad \Longrightarrow \quad f(x_n) \to b.$$

The proof of the following theorem is left as an exercise.

Theorem 2.10 *Let f be a real valued function defined on a set $D \subseteq \mathbb{R}$, and let $a \in \mathbb{R}$ be a limit point of D. Then $\lim_{x \to a} f(x)$ exists if and only if $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ exist and $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$, and in that case*

$$\lim_{x \to a} f(x) = \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x).$$

In view of the above theorem, if $\lim_{x \to a^-} f(x)$ does not exist or $\lim_{x \to a^+} f(x)$ does not exist or both $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ exist but $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$, then $\lim_{x \to a} f(x)$ does not exist.

Example 2.10 Let us consider the a few examples to illustrate Theorem 2.10.

(i) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1/x, & x > 0, \\ 1, & x \leq 0. \end{cases}$$

In this case we see that $\lim_{x \to 0^-} f(x) = 1$, but $\lim_{x \to 0^+} f(x)$ does not exist.

(ii) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1/x, & x < 0, \\ 1, & x \geq 0. \end{cases}$$

In this case we see that $\lim_{x\to 0^+} f(x) = 1$, but $\lim_{x\to 0^-} f(x)$ does not exist.

(ii) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1/x, & x \neq 0, \\ 1, & x = 0. \end{cases}$$

In this case, both $\lim_{x\to 0^+} f(x)$ and $\lim_{x\to 0^-} f(x)$ do not exist.

(iii)

Let f be as in Example 2.3, that is, $f : [-1, 1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0, & -1 \le x \le 0, \\ 1, & 0 < x \le 1. \end{cases}$$

In this case both $\lim_{x\to 0^-} f(x)$ and $\lim_{x\to 0^+} f(x)$ exist, but $\lim_{x\to 0} f(x)$ does not exist. □

2.1.6 Limit at ∞ and at $-\infty$

Definition 2.5 Suppose a function f is defined on an interval of the form (a, ∞) for some $a \in \mathbb{R}$. Then we say that $f(x)$ **has the limit** b **as** $x \to \infty$, if for every $\varepsilon > 0$, there exits $M > a$ such that

$$|f(x) - b| < \varepsilon \quad \text{whenever} \quad x > M,$$

and in that case we write $\lim_{x\to\infty} f(x) = b$ □

Definition 2.6 Suppose a function f is defined on an interval of the form $(-\infty, a)$ for some $a \in \mathbb{R}$. Then we say that $f(x)$ **has the limit** b **as** $x \to -\infty$, if for every $\varepsilon > 0$, there exits $M < a$ such that

$$|f(x) - b| < \varepsilon \quad \text{whenever} \quad x < M,$$

and in that case we $\lim_{x\to -\infty} f(x) = b$, □

The following can be verified by applying the corresponding results for sequences (Verify).

1. If $\lim_{x\to\infty} f(x) = b$ and $\lim_{x\to\infty} g(x) = c$, then

$$\lim_{x\to\infty} [f(x) + g(x)] = b + c, \qquad \lim_{x\to\infty} f(x)g(x) = bc.$$

2. If If $\lim_{x\to\infty} f(x) = b$, $\lim_{x\to\infty} g(x) = c$ and $c \not0$, then there exists $M_0 > 0$ such that $g(x) \neq 0$ for all $x > M_0$ and

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \frac{b}{c}.$$

Example 2.11 (i) We show that $\lim_{x\to\infty} \frac{1}{x} = 0$. Taking $f(x) = \frac{1}{x}$ for $x \neq 0$, $b = 0$ and $\varepsilon > 0$, we observe that

$$|f(x) - b| < \varepsilon \iff \frac{1}{|x|} < \varepsilon \iff |x| > \frac{1}{\varepsilon}.$$

Hence,

$$x > 1/\varepsilon \quad \Longrightarrow \quad |x| > 1/\varepsilon \quad \Longrightarrow \quad |f(x) - b| < \varepsilon.$$

This shows that $|f(x) - b| < \varepsilon$ whenever $x > M := 1/\varepsilon$.

(ii) We show that $\lim_{x\to-\infty} \frac{1}{x} = 0$. As before, taking $f(x) = \frac{1}{x}$ for $x \neq 0$, $b = 0$ and $\varepsilon > 0$, we observe that

$$|f(x) - b| < \varepsilon \iff \frac{1}{|x|} < \varepsilon \iff |x| > \frac{1}{\varepsilon}.$$

Hence,

$$x < -1/\varepsilon \quad \Longrightarrow \quad |x| > 1/\varepsilon \quad \Longrightarrow \quad |f(x) - b| < \varepsilon.$$

This shows that $|f(x) - b| < \varepsilon$ whenever $x < M := -1/\varepsilon$.

(iii) We show that $\lim_{x\to\infty} \frac{1}{x^2} = 0$. Taking $f(x) = \frac{1}{x^2}$ for $x \neq 0$, $b = 0$ and $\varepsilon > 0$, we observe that

$$|f(x) - b| < \varepsilon \iff \frac{1}{x^2} < \varepsilon \iff |x| > \frac{1}{\sqrt{\varepsilon}}.$$

Hence,

$$x > 1/\sqrt{\varepsilon} \quad \Longrightarrow \quad |x| > 1/\sqrt{\varepsilon} \quad \Longrightarrow \quad |f(x) - b| < \varepsilon.$$

This shows that $|f(x) - b| < \varepsilon$ whenever $x > M := 1/\sqrt{\varepsilon}$.

(iv) We show that $\lim_{x\to\infty} \frac{1+x}{1+x^2} = 0$. Let $f(x) = \frac{1+x}{1+x^2}$ for $x \in \mathbb{R}$. The, by (i) and (iii) above,

$$f(x) = \frac{1+x}{1+x^2} = \frac{1/x^2 + 1/x}{1/x^2 + 1} \to \frac{0}{1} = 0.$$

(v) We show that $\lim_{x\to\infty} \frac{1+x}{1-x} = -1$. Let $f(x) = \frac{1+x}{1-x}$ for $x \neq 1$. By (i) above,

$$f(x) = \frac{1+x}{1-x} = \frac{1/x + 1}{1/x - 1} \to \frac{1}{-1} = -1.$$

(vi) We show that $\lim_{x\to\infty} \frac{1+2x}{1+3x} = \frac{2}{3}$. Let $f(x) = \frac{1+2x}{1+3x}$ for $x \neq -1/3$. Then, by (i),

$$f(x) = \frac{1+2x}{1+3x} = \frac{1/x + 2}{1/x + 3} = \frac{2}{3}.$$

□

Definition 2.7 We define the following:

1. $\lim_{x \to a} f(x) = \infty$ if for every $M > 0$, there exists $\delta > 0$ such that
$$0 < |x - a| < \delta \Longrightarrow f(x) > M.$$

2. $\lim_{x \to a} f(x) = -\infty$ if for every $M > 0$, there exists $\delta > 0$ such that
$$0 < |x - a| < \delta \Longrightarrow f(x) < -M.$$

3. $\lim_{x \to +\infty} f(x) = \infty$ if for every $M > 0$, there exists $\alpha > 0$ such that
$$x > \alpha \Longrightarrow f(x) > M.$$

4. $\lim_{x \to +\infty} f(x) = -\infty$ if for every $M > 0$, there exists $\alpha > 0$ such that
$$x > \alpha \Longrightarrow f(x) < -M.$$

5. $\lim_{x \to -\infty} f(x) = \infty$ if for every $M > 0$, there exists $\alpha > 0$ such that
$$x < -\alpha \Longrightarrow f(x) > M.$$

6. $\lim_{x \to -\infty} f(x) = -\infty$ if for every $M > 0$, there exists $\alpha > 0$ such that
$$x < -\alpha \Longrightarrow f(x) < -M.$$

□

It can be easily shown (*Verify*) that

$$\lim_{x \to a} f(x) = \infty \iff \lim_{x \to a} [-f(x)] = -\infty,$$
$$\lim_{x \to +\infty} f(x) = \infty \iff \lim_{x \to +\infty} [-f(x)] = -\infty,$$
$$\lim_{x \to -\infty} f(x) = \infty \iff \lim_{x \to -\infty} [-f(x)] = -\infty.$$

Example 2.12 (i) We show that $\lim_{x \to 0} \frac{1}{x^2} = \infty$.

Taking $f(x) = \frac{1}{x^2}$ for $x \neq 0$ and $M > 0$, we observe that

$$f(x) > M \iff \frac{1}{x^2} > M \iff |x| < \frac{1}{\sqrt{M}}.$$

Hence, for $0 < \delta < 1/\sqrt{M}$,

$$|x| < \delta \quad \Longrightarrow \quad |x| < \frac{1}{\sqrt{M}} \quad \Longrightarrow \quad f(x) > M.$$

Thus, $\lim_{x\to 0} \frac{1}{x^2} = \infty$.

(ii) We show that $\lim_{x\to 1} \left|\frac{1+x}{1-x}\right| = \infty$.

Let $f(x) = \left|\frac{1+x}{1-x}\right|$ for $x \neq 1$. Then for $M > 0$,

$$f(x) = \left|\frac{1+x}{1-x}\right| > M \iff |1-x| < \frac{|1+x|}{M}$$

and

$$|1+x| = |2-(1-x)| \geq 2 - |1-x| > 1 \quad \text{whenever} \quad |x-1| < 1.$$

Hence

$$|x-1| < 1 \quad \text{and} \quad |x-1| < \frac{1}{M} \quad \Longrightarrow \quad |1-x| < \frac{|1+x|}{M} \quad \Longrightarrow \quad f(x) > M$$

Thus,

$$|x-1| < \delta := \min\{1, 1/M\} \quad \Longrightarrow \quad f(x) > M$$

showing that $\lim_{x\to 1} \left|\frac{1+x}{1-x}\right| = \infty$.

(iii) Let $f(x) = x^2$, $x \in \mathbb{R}$. We show that $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to -\infty} f(x) = \infty$.

For $M > 0$,

$$f(x) = x^2 > M \iff |x| > \sqrt{M}.$$

Thus,

$$x > \sqrt{M} \quad \Longrightarrow f(x) > M$$

and

$$x < -\sqrt{M} \quad \Longrightarrow f(x) > M.$$

□

2.2 Continuity of a Function

In this section we assume that the domain of a real valued function is an interval I. Recall that every point in an interval I is a limit point of I.

2.2.1 Definition and some basic results

Definition 2.8 Let f be a real valued function defined on an interval I.

1. The function f is said to be **continuous at** $a \in I$ if $\lim_{x \to a} f(x) = f(a)$.
2. The function f is said to be **continuous on** I if f is continuous at every $a \in I$.

□

Using the definition of limit, it follows that

- f is continuous at $a \in I$ if and only if for every $\varepsilon > 0$, there exists $\delta > 0$ such that
$$|f(x) - f(a)| < \varepsilon \quad \text{whenever} \quad x \in I,\ |x - a| < \delta.$$

CONVENTION: Suppose the domain of a function f is not specified explicitly. Even then we may say that f is continuous at a point $a \in \mathbb{R}$ to mean that f is defined on an interval containing a and f is continuous at a.

By Theorems 2.3 and 2.4, we have the following.

Theorem 2.11 *A function $f : I \to \mathbb{R}$ is continuous at $a \in I$ if and only if for every sequence (x_n) in I with $x_n \to a$, we have $f(x_n) \to f(a)$.*

Also, using Theorem 2.5, we obtain the following.

Theorem 2.12 *Suppose f and g are defined on an interval I and both f and g are continuous at $a \in I$. Then we have the following.*

(i) *$f + g$ and fg are continuous at a.*

(ii) *If $g(a) \neq 0$, then there exists $\delta_0 > 0$ such that $g(x) \neq 0$ for every x in the interval $I_0 := I \cap (a - \delta, a + \delta)$, and the function f/g which is defined on I_0 is continuous at a.*

From Theorem 2.9, we have the following.

Theorem 2.13 *Suppose f is continuous at a point a and g is continuous at the point $b := f(a)$. Then $g \circ f$ is continuous at a.*

The following property of a continuous function is worth noticing.

Theorem 2.14 *Suppose f is a continuous function defined on an interval I and $a \in I$. If $\alpha \in \mathbb{R}$ is such that $f(a) > \alpha$, then there exists $\delta > 0$ such that*

$$f(x) > \alpha \quad \forall\, x \in I \cap (a - \delta, a + \delta).$$

Proof. Since f is continuous at a, for any $\varepsilon > 0$, there exists $\delta > 0$ (depending on ε) such that

$$x \in (a - \delta, a + \delta) \cap I \quad \Longrightarrow \quad f(x) \in (f(a) - \varepsilon, f(a) + \varepsilon).$$

In particular,

$$x \in (x_0 - \delta, x_0 + \delta) \cap I \quad \Longrightarrow \quad f(x) > f(a) - \varepsilon.$$

Thus, taking $0 < \varepsilon < f(a) - \alpha$, we have $f(x) > \alpha$ for all $x \in (x_0 - \delta, x_0 + \delta) \cap I$. ∎

Proof using sequences. Suppose the the conclusion in the theorem does not hold. Then for each $n \in \mathbb{N}$, $x_n \in I \cap (a - 1/n, a + 1/n)$ such that $f(x_n) \leq \alpha$. Thus, we have $x_n \to a$ as $n \to \infty$, and hence, by Theorem 2.11, $f(x_n) \to f(a)$. Since $f(x_n) \leq \alpha$ for all $n \in \mathbb{N}$, we have $f(a) \leq \alpha$, which contradicts the assumption that $f(a) > \alpha$. ∎

Corollary 2.15 *Suppose f is a continuous function defined on an interval I and $a \in I$. If $f(a) > 0$, then there exists an interval $I_0 \subseteq I$ containing a such that $f(x) > 0$ for every $x \in I_0$.*

Example 2.13 If $f(x)$ is a polynomial, say $f(x) = a_0 + a_1 x + \ldots + a_k x^k$, then f is continuous on $\mathbb{R}$ □

Example 2.14 For given $x_0 \in \mathbb{R}$, let $f(x) = |x - x_0|$, $x \in \mathbb{R}$. Then f is continuous on $\mathbb{R}$. To see this, note that, for $a \in \mathbb{R}$,

$$|f(x) - f(a)| = ||x - x_0| - |a - x_0|| \leq |(x - x_0) - (a - x_0)| = |x - a|.$$

Hence, for every $\varepsilon > 0$, we have

$$|x - a| < \varepsilon \Longrightarrow |f(x) - f(a)| < \varepsilon.$$

□

Example 2.15 Let $f(x) = \frac{x^2 - 4}{x - 2}$ for $x \in \mathbb{R} \setminus \{2\}$ and $f(2) = 4$. Then f is continuous on $\mathbb{R}$ (*Verify*). □

Example 2.16 The functions f and g defined by $f(x) = \sin x$ and $g(x) = \cos x$ for $x \in \mathbb{R}$ are continuous on $\mathbb{R}$:

Note that for $x, y \in \mathbb{R}$,

$$\sin x - \sin y = 2 \sin\left(\frac{x - y}{2}\right) \cos\left(\frac{x + y}{2}\right)$$

so that

$$|\sin x - \sin y| \le |x - y| \qquad \forall x, y \in \mathbb{R}.$$

Hence, for every $\varepsilon > 0$ and for every $x_0 \in \mathbb{R}$,

$$|x - x_0| < \varepsilon \Longrightarrow |\sin x - \sin x_0| < \varepsilon.$$

Thus, $f(x) = \sin x$ is continuous at every $x \in \mathbb{R}$. Since $\cos x = 1 - 2\sin^2(x/2)$, $x \in \mathbb{R}$, it also follows that $g(x) = \cos x$ is continuous at every $x \in \mathbb{R}$. $\square$

Example 2.17 Let $f(x) = \frac{\sin x}{x}$ for $x \neq 0$ and $f(0) = 1$. Then f is continuous at every point in $\mathbb{R}$ (*Verify*). $\square$

Example 2.18 The function f defined by $f(x) = 1/x$, $x \neq 0$ is continuous at every $x_0 \neq 0$:

Note that for $x \neq 0$, $x_0 \neq 0$,

$$\left|\frac{1}{x} - \frac{1}{x_0}\right| = \frac{|x - x_0|}{|x x_0|} \le \frac{2|x - x_0|}{|x_0^2|} \quad \text{whenever} \quad |x - x_0| \le \frac{|x_0|}{2}$$

since $|x| = |x_0 - (x_0 - x)| \ge |x_0| - |x_0 - x|$. Hence, for any $\varepsilon > 0$,

$$\left|\frac{1}{x} - \frac{1}{x_0}\right| < \varepsilon \quad \text{whenever} \quad |x - x_0| < \delta := \min\{\frac{\varepsilon x_0^2}{2}, \frac{x_0}{2}\}.$$

Thus, f is continuous at every $x_0 \neq 0$. $\square$

Example 2.19 Let f be defined by $f(x) = 1/x$ on $(0, 1]$. Then there does not exist a continuous function g on $[0, 1]$ such that $g(x) = f(x)$ for all $x \in (0, 1]$:

Suppose g is any function defined on $[0, 1]$ such that $g(x) = f(x)$ for all $x \in (0, 1]$. Then we have $1/n \to 0$ but $g(1/n) = f(1/n) = n \to \infty$. Thus, $g(1/n) \not\to g(0)$. $\square$

Example 2.20 The function f defined by $f(x) = \sqrt{x}$, $x \ge 0$ is continuous at every $x_0 \ge 0$:

Let $\varepsilon > 0$ be given. First consider the point $x_0 = 0$. Then we have

$$|f(x) - f(x_0)| = \sqrt{x} < \varepsilon \quad \text{whenever} \quad |x| < \varepsilon^2.$$

Thus, f is continuous at $x_0 = 0$. Next assume that $x_0 > 0$. Since $|x - x_0| = (\sqrt{x} + \sqrt{x_0})|\sqrt{x} - \sqrt{x_0}|$, we have

$$|\sqrt{x} - \sqrt{x_0}| = \frac{|x - x_0|}{\sqrt{x} + \sqrt{x_0}} \le \frac{|x - x_0|}{\sqrt{x_0}}.$$

Thus,

$$|\sqrt{x} - \sqrt{x_0}| < \varepsilon \quad \text{whenever} \quad |x - x_0| < \delta := \varepsilon\sqrt{x_0}.$$

$\square$

More generally, we have the following example.

Example 2.21 Let $k \in \mathbb{N}$. Then the function f defined by $f(x) = x^{1/k}$, $x \geq 0$ is continuous at every $x_0 \geq 0$:

Let $\varepsilon > 0$ be given. First consider the point $x_0 = 0$. Then we have

$$|f(x) - f(x_0)| = x^{1/k} < \varepsilon \quad \text{whenever} \quad |x| < \varepsilon^k.$$

Thus, f is continuous at $x_0 = 0$. Next assume that $x_0 > 0$. Let $y = x^{1/k}$ and $y_0 = x_0^{1/k}$. Since

$$y^k - y_0^k = (y - y_0)(y^{k-1} + y^{k-2}y_0 + \ldots + yy^{k-2} + y_0^{k-1}),$$

so that

$$x - x_0 = (x^{1/k} - x_0^{1/k})(y^{k-1} + y^{k-2}y_0 + \ldots + yy^{k-2} + y_0^{k-1}).$$

Hence,

$$|x^{1/k} - x_0^{1/k}| = \frac{|x - x_0|}{y^{k-1} + y^{k-2}y_0 + \ldots + yy^{k-2} + y_0^{k-1}} \leq \frac{|x - x_0|}{y_0^{k-1}}.$$

Thus,

$$|x^{1/k} - x_0^{1/k}| < \varepsilon \quad \text{whenever} \quad |x - x_0| < \delta := \varepsilon y_0^{k-1} = \varepsilon x_0^{1-1/k}.$$

Thus, f is continuous at every $x_0 > 0$. □

Example 2.22 For a rational number r, let $f(x) = x^r$ for $x > 0$. Then using Example 2.21 together with Theorem 2.13, we see that f is continuous at every $x_0 > 0$. □

We know that given $r \in \mathbb{R}$, there exists a sequence (r_n) of rational numbers such that $r_n \to r$. For $n \in \mathbb{N}$, let $f_n(x) = x^{r_n}$, $x > 0$. Since each f_n is continuous for $x > 0$, one may enquire whether the function f defined by $f(x) = x^r$ is continuous for $x > 0$.

First of all how do we define the x^r for $x > 0$?

We shall discuss this issue in the next subsection, where we shall introduce two important classes of functions, namely, *exponential* and *logarithm functions.* In fact, our discussion will also include, as special cases, the Examples 2.20 - 2.22.

Exercise 2.8 Let I be an interval and $f : I \to \mathbb{R}$. Suppose there exists a constant $K > 0$ such that

$$|f(x) - f(y)| \leq K|x - y| \quad \forall\, x, y \in I. \tag{$*$}$$

Show that f is continuous on I. Find an example of a continuous function which does not satisfy $(*)$ for any $K > 0$. [*Hint:* Consider $f(x) = \frac{1}{x}$ for $x \in (0, 1]$.]

A function f satisfying $(*)$ for some $K > 0$ is called a *Lipschitz continuous function*, and the constant $K0$ called the *Lipschitz constant.* ◄

2.2.2 Exponential and logarithm functions

We have already come across expression such as a^b for $a > 0$ and $b \in \mathbb{R}$, though we have not proved existence of such numbers, and also defined a number denoted by e as the limit of the sequence $\left(1 + \frac{1}{n}\right)^{1/n}$ or the series $\sum_{n=0}^{\infty} \frac{1}{n!}$. Now, we formally define the following functions:

- **Exponential function**: e^x, $x \in \mathbb{R}$.
- **Natural logarithm** function: $\ln x$, $x > 0$.
- **Exponential function**: a^x, $x \in \mathbb{R}$ for a given $a > 0$.
- **Logarithm function** with base $a > 0$: $\log_a x$, $x > 0$.

First, we observe that for every $x \in \mathbb{R}$, the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges absolutely. This can be seen by using the ratio test. This series plays a very significant role in mathematics.

Definition 2.9 For $x \in \mathbb{R}$, let

$$\exp(x) := \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

The function $\exp(x)$, $x \in \mathbb{R}$, is called the **exponential function**. □

From the above definition, it is clear that

$$\exp(0) = 1 \quad \text{and} \quad \exp(1) = e.$$

In order to derive some of the important properties of the function $\exp(x)$, the student is urged to do the following exercise.

Exercise 2.9 Suppose that series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are absolutely convergent. Then, the series

$$\sum_{n=0}^{\infty} c_n \quad \text{with} \quad c_n := \sum_{k=0}^{n} a_k b_{n-k} \qquad (*)$$

is absolutely convergent. Further, show that if $\alpha = \sum_{n=0}^{\infty} a_n$ and $\beta = \sum_{n=0}^{\infty} b_n$, then $\sum_{n=0}^{\infty} c_n = \alpha\beta$.

The series $\sum_{n=0}^{\infty} c_n$ defined in $(*)$ is called the **Cauchy product** of the series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$. ◀

Using the conclusion in the above exercise, it can be proved that

$$\exp(x + y) = \exp(x)\exp(y) \qquad \forall\, x, y \in \mathbb{R}. \qquad (**)$$

Exercise 2.10 Prove $(**)$ above. ◀

We observe the following properties:

- $\exp(-x) = \dfrac{1}{\exp(x)}$ $\forall\, x \in \mathbb{R}$. In particular, since $\exp(x) > 0$ for $x \geq 0$, we

have

$$\exp(x) > 0 \qquad \forall\, x \in \mathbb{R}.$$

[This follows from $(**)$]

- $\exp(x) > 1 \iff x > 0$ and $\exp(x) = 1 \iff x = 0$.

[From the definition, $x > 0$ implies $\exp(x) > 1$. Next, suppose $x \leq 0$. If $x = 0$, then $\exp(x) = \exp(0) = 1$. If $x < 1$, then taking $y = -x$, we have $y > 1$, and hence from the first part, $\exp(y) > 1$, i.e., $1/\exp(x) = \exp(-x) > 1$ so that $\exp(x) < 1$. Hence, $\exp(x) > 1 \iff x > 0$. From this, we get $\exp(x) = 1 \iff x = 0$.]

- $x > y \iff \exp(x) > \exp(y)$.

[$x > y \iff x - y > 0 \iff \exp(x - y) > 1$. But, $\exp(x - y) = \exp(x)/\exp(y)$.]

- $\exp(kx) = [\exp(x)]^k$ $\forall\, x \in \mathbb{R},\ k \in \mathbb{Z}$. In particular, taking $x = 1$ and $x = -1$,

$$\exp(k) = e^k \qquad \forall\, k \in \mathbb{Z}.$$

- Since $e = \exp(1) = \exp(k/k) = [\exp(1/k]^k$ $\forall\, k \in \mathbb{N}$, we have

$$\exp(1/k) = e^{1/k} \qquad \forall\, k \in \mathbb{N}.$$

- $\exp(m/n) = e^{m/n}$ $\forall\, m, n \in \mathbb{N}$. Hence,

$$\exp(r) = e^r \qquad \forall\, r \in \mathbb{Q}.$$

We know that every real number is a limit of a sequence of rational numbers. Thus, if $r \in \mathbb{R}$, there exits a sequence (r_n) of rational numbers that $r_n \to r$. So, it is natural to define

$$e^r = \lim_{n\to\infty} e^{r_n}$$

provided the above limit exists. Thus, our next attempt is to show that the function $\exp(x)$, $x \in \mathbb{R}$, is continuous.

In view of these observations, we use the following

Proposition 2.16 *The following results hold.*

(i) $\exp(x) \to \infty$ *as* $x \to \infty$.

(ii) $\exp(x) \to 0$ *as* $x \to -\infty$.

Proof. For $x \geq 0$, we have

$$\exp(x) = 1 + x + \sum_{n=2}^{\infty} \frac{x^n}{n!} \geq 1 + x.$$

From this (i) follow. To see (ii), let $x < 0$ and $y = -x$. Then $y > 0$ so that by (ii),

$$\exp(x) = \exp(-y) = \frac{1}{\exp(y)} \to 0 \quad \text{as} \quad y \to \infty.$$

Thus, $\exp(x) \to 0$ as $x \to -\infty$. ▮

NOTATION: For brevity of expression, we shall use the notation e^x for $\exp(x)$.

Theorem 2.17 *The function* $\exp(x)$ *is continuous on* $\mathbb{R}$

Proof. Let $x, x_0 \in R$. Then we have

$$e^x - e^{x_0} = e^{x_0}(e^{x-x_0} - 1) = e^{x_0} \sum_{n=1}^{\infty} \frac{(x-x_0)^n}{n!} = e^{x_0}(x - x_0) \sum_{n=1}^{\infty} \frac{(x-x_0)^{n-1}}{n!}.$$

Thus, if $|x - x_0| \leq 1$, then

$$|e^x - e^{x_0}| \leq e^{x_0}|x - x_0| \sum_{n=1}^{\infty} \frac{1}{n!} = e^{x_0}(e-1)|x - x_0|.$$

Hence, for every $\varepsilon > 0$,

$$|e^x - e^{x_0}| < \varepsilon \quad \text{whenever} \quad |x - x_0| < \min\{1,\ \varepsilon/[e^{x_0}(e-1)]\}$$

so that e^x is a continuous function for $x \in \mathbb{R}$. ▮

Theorem 2.18 *The function* e^x *is bijective from* $\mathbb{R}$ *to* $(0, \infty)$.

Proof. First we observe that, for x_1, x_2 in $\mathbb{R}$

$$e^{x_2} - e^{x_1} = e^{x_1}[e^{x_2 - x_1} - 1].$$

Thus,

$$e^{x_2} = e^{x_1} \iff e^{x_2 - x_1} = 1 \iff x_1 = x_2,$$

showing that the function $x \mapsto e^x$ is one-one.

Next, to show the function is onto, let $y \in (0, \infty)$. Since, by Proposition 2.16,

$$e^x \to 0 \quad \text{as} \quad x \to -\infty, \qquad e^x \to \infty \quad \text{as} \quad x \to \infty,$$

by intermediate value property, there exists $x \in \mathbb{R}$ such that $e^x = y$. ▮

Definition 2.10 For $b > 0$, the unique $a \in \mathbb{R}$ such that $e^a = b$ is called the **natural logarithm** of b, and it is denoted by $\ln b$. The function

$$\ln x, \qquad x > 0,$$

is called the **natural logarithm function**. □

Definition 2.11 For $a > 0$ and $b \in \mathbb{R}$, we define

$$a^b := e^{b \ln a}.$$

□

Remark 2.3 We note that $\ln e = 1$ so that if $a = e$, then the Definition 2.11 matches with Definition 2.9. ♦

Theorem 2.19 *Let $a > 0$. Then the function a^x is continuous and bijective from $\mathbb{R}$ to $(0, \infty)$.*

Proof. Note that for $x \in \mathbb{R}$, $a^x := e^{x \ln a}$. Hence, the result is a consequence of Theorems 2.17 and 2.18, and the Definition 2.11, and using the fact that composition of two continuous functions is continuous. ▮

Definition 2.12 Let $a > 0$. For $c > 0$, the unique $b \in \mathbb{R}$ such that $a^b = c$ is called the **logarithm** of c to the base a, and it is denoted by $\log_a c$. The function

$$\log_a x, \qquad x > 0,$$

is called the **logarithm function**. □

We observe that following.

- For $y \in \mathbb{R}$, $\quad y = \ln x \iff e^y = x$.
- For $a > 0$ and $y \in \mathbb{R}$, $\quad y = \log_a x \iff a^y = x$.
- For $a > 0$ and $x > 0$, $\quad \log_a x = \dfrac{\ln x}{\ln a}$.

Exercise 2.11 For $a > 0, b > 0$, show that $\log_b a \log_a b = 1$. ◀

Theorem 2.20 *The functions $\ln x$ and $\log_a x$ for $a > 0$ are continuous on $(0, \infty)$.*

Proof. Let x, x_0 belong to the interval $(0, \infty)$, and let $y = \ln x$ and $y_0 = \ln x_0$. Then we have $e^y = x$ and $e^{y_0} = x_0$. Assume, without loss of generality that $x > x_0$. Since $e^a > 1$ if and only if $a > 0$, we have $y > y_0$, and hence

$$x - x_0 = e^y - e^{y_0} = e^{y_0}(e^{y - y_0} - 1) = e^{y_0} \sum_{n=1}^{\infty} \frac{(y - y_0)^n}{n!} \geq e^{y_0}(y - y_0).$$

Hence,

$$|y - y_0| \leq e^{-y_0}|x - x_0|.$$

Thus, for $\varepsilon > 0$, we have $|y - y_0| < \varepsilon$ whenever $|x - x_0| < e^{y_0}\varepsilon$, $\ln x$ is continuous on $(0, \infty)$. Since $\log_a x = \ln x / \ln a$, the function $\log_a x$ is also continuous on $(0, \infty)$. ▮

Theorem 2.21 *For $r \in \mathbb{R}$, the function $f : (0, \infty) \to \mathbb{R}$ be defined by*

$$f(x) = x^r, \qquad x \in (0, \infty)$$

is continuous.

Proof. For $r \in \mathbb{R}$ and $x > 0$, we have $x^r = e^{r \ln x}$. Hence, the result follows from Theorem 2.20 and Theorem 2.13. ∎

Remark 2.4 Often, the notation $\log x$ is used for the natural logarithm function instead of $\ln x$. ♦

Example 2.23 Recall that $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$ exists, and we denoted it by e. Now we show that

$$\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x = e.$$

Let $\varepsilon > 0$ be given. We have to find an $M > 0 \in \mathbb{N}$ such that

$$e - \varepsilon < \left(1 + \frac{1}{x}\right)^x < e + \varepsilon \quad \text{whenever} \quad x > M. \tag{$*$}$$

Now, we can see that, for every $n \in \mathbb{N}$, if $x \in \mathbb{R}$ is such that $n \le x \le n+1$, then

$$1 + \frac{1}{n+1} \le 1 + \frac{1}{x} \le 1 + \frac{1}{n}$$

so that

$$\left(1 + \frac{1}{n+1}\right)^n \le \left(1 + \frac{1}{x}\right)^x \le \left(1 + \frac{1}{n}\right)^{n+1}.$$

Thus is is same as

$$\alpha_n \le \left(1 + \frac{1}{x}\right)^x \le \beta_n,$$

where

$$\alpha_n := \left(1 + \frac{1}{n+1}\right)^{-1} \left(1 + \frac{1}{n+1}\right)^{n+1}, \qquad \beta_n := \left(1 + \frac{1}{n}\right)^n \left(1 + \frac{1}{n}\right).$$

We know that $\alpha_n \to e$ and $\beta_n \to e$ as $n \to \infty$. Therefore, there exists $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$,

$$e - \varepsilon < \alpha_n < e + \varepsilon, \qquad e - \varepsilon < \beta_n < e + \varepsilon.$$

Now, take $M = n_0$ and let $x > M$. Take $n \ge n_0$ such that $n + 1 \ge x \ge n$. For this n and x, we have

$$e - \varepsilon < \alpha_n \le \left(1 + \frac{1}{x}\right)^x \le \beta_n < e + \varepsilon.$$

Thus, we obtained an $M > 0$ such that

$$e - \varepsilon < \left(1 + \frac{1}{x}\right)^x < e + \varepsilon \quad \text{whenever} \quad x > M.$$

Thus, we have proved $(*)$. □

Using the arguments used in the above example, we obtain a more general result.

Theorem 2.22 *Suppose (α_n) and (β_n) are sequences of positive real numbers and f is a (real valued) function defined on $(0, \infty)$ having the following property: For $n \in \mathbb{N}$, $x \in \mathbb{R}$,*

$$n < x < n + 1 \implies \alpha_n \le f(x) \le \beta_n.$$

If (α_n) and (β_n) converge to the same limit, say b, then $\lim_{x\to\infty} f(x) = b$.

2.2.3 Some properties of continuous functions

Recall that a subset S of $\mathbb{R}$ is said to be *bounded* if there exists $M > 0$ such that $|s| \le M$ for all $s \in S$, and set which is not bounded is called an *unbounded set.*

Recall that if S is a bounded subset of $\mathbb{R}$, then S has infimum and supremum.

Exercise 2.12 Let $S \subseteq \mathbb{R}$. Prove the following:

(i) Suppose S is bounded, and say $\alpha := \inf S$ and $\beta := \sup S$. Then there exist sequences (s_n) and (t_n) in S such that $s_n \to \alpha$ and $t_n \to \beta$.

(ii) S is unbounded if and only if there exists a sequence (s_n) in S which is unbounded.

(iii) S is unbounded if and only if there exists a sequence (s_n) in S such that $|s_n| \to \infty$ as $n \to \infty$.

(iv) If (s_n) is a sequence in S which is unbounded, then there exists a subsequence (s_{k_n}) of (s_n) such that $|s_{k_n}| \to \infty$ as $n \to \infty$.

(v) If (s_n) is a sequence in S such that $|s_n| \to \infty$ as $n \to \infty$, and if (s_{k_n}) is subsequence of (s_n), then $|s_{k_n}| \to \infty$ as $n \to \infty$. ◀

Definition 2.13 A real valued function defined on a set $D \subseteq \mathbb{R}$ is said to be a **bounded function** if the set $\{f(x) : x \in D\}$ is bounded. A function is said to be an **unbounded function** if it is not bounded. □

The following can be easily deduced from the definition:

- A function $f : D \to \mathbb{R}$ is bounded if and only if there exists $M > 0$ such that $|f(x)| \le M$ for all $x \in D$.
- A function $f : D \to \mathbb{R}$ is unbounded if and only if there exists a sequence $(x_n) \in D$ such that the $|f(x_n)| \to \infty$ as $n \to \infty$.

Theorem 2.23 *Suppose f is a real valued function defined on a closed and bounded interval $[a, b]$. Then f is a bounded function.*

Proof. Assume for the time being that f is not a bounded function. Then, there exists a sequence (x_n) in $[a, b]$ such that $|f(x_n)| \to \infty$ as $n \to \infty$. Since

(x_n) is a bounded sequence, by Bolzano-Weierstrass property of $\mathbb{R}$, there exists a subsequence (x_{k_n}) of (x_n) such that $x_{k_n} \to x$ for some $x \in [a, b]$. Therefore, by continuity of f, $f(x_{k_n}) \to f(x)$. In particular, $(f(x_{k_n}))$ is a bounded sequence. This is a contradiction to the fact that $|f(x_n)| \to \infty$ as $n \to \infty$. Thus, we have proved that f cannot be unbounded. ∎

Attaining $\max f$ **and** $\min f$

Theorem 2.24 *Suppose f is a continuous real valued function defined on a closed and bounded interval $[a, b]$. Then there exists x_0, y_0 in $[a, b]$ such that*

$$f(x_0) = \alpha := \inf\{f(x) : x \in [a, b]\}, \qquad f(y_0) = \beta := \sup\{f(x) : x \in [a, b]\}.$$

Proof. By the definition of infimum and supremum, there exist sequences (x_n) and (y_n) in $[a, b]$ such that $f(x_n) \to \alpha$ and $f(y_n) \to \beta$ as $n \to \infty$. Since (x_n) and (y_n) are bounded sequences, there exist subsequences (x_{k_n}) and (y_{k_n}) of (x_n) and (y_n), respectively, such that $x_{k_n} \to x$ and $y_{k_n} \to y$ for some x, y in $[a, b]$. By continuity of f, $f(x_{k_n}) \to f(x)$ and $f(y_{k_n}) \to f(y)$ as $n \to \infty$. But, we already have $f(x_{k_n}) \to \alpha$ and $f(y_{k_n}) \to \beta$. Hence, $\alpha = f(x)$ and $\beta = f(y)$. ∎

Remark 2.5 By Theorem 2.24, we say that the infimum and supremum of a continuous real valued function f defined on a closed and bounded interval $[a, b]$ are attained at some points in $[a, b]$, and in that case, we write

$$\inf\{f(x) : x \in [a, b]\} = \min_{a \le x \le b} f(x), \qquad \sup\{f(x) : x \in [a, b]\} = \max_{a \le x \le b} f(x).$$

The conclusion in the above theorem need hold if the domain of the function is not of the form $[a, b]$ or if f is not continuous. For example, $f : (0, 1] \to \mathbb{R}$ defined by $f(x) = 1/x$ for $x \in (0, 1]$ is continuous, but does not attain supremum. Same is the case if $g : [0, 1] \to \mathbb{R}$ is defined by

$$g(x) = \begin{cases} \frac{1}{x}, & x \in (0, 1], \\ 1, & x = 0. \end{cases}$$

Thus, neither continuity nor the fact that the domain is a closed and bounded interval can be dropped. This does not mean that the conclusion in the theorem does not hold for all such functions! For example $f : [0, 1) \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0, & x \in [0, 1/2), \\ 1, & x \in [1/2, 1). \end{cases}$$

Then we see that neither f is continuous, nor its domain of the form $[a, b]$. But, f attains both its maximum and minimum. ♦

Intermediate value theorem

Suppose f is a continuous real valued function defined on a closed and bounded interval $[a, b]$, and

$$\alpha := \min_{a \le x \le b} f(x), \qquad \beta := \max_{a \le x \le b} f(x).$$

Clearly,

$$\alpha \le f(x) \le \beta \qquad \forall\, x \in [a, b].$$

Now, the question is whether every value between α and β is attained by the function. The answer is in affirmative. In fact we have the following general theorem, known as *Intermediate value theorem*[1]

Theorem 2.25 (Intermediate value theorem) *Suppose f is a continuous real valued function defined on an interval I. Suppose x_1 and x_2 are in I such that $f(x_1) < f(x_2)$, and c is such that $f(x_1) < c < f(x_2)$. Then there exists x_0 lying between x_1 and x_2 such that $f(x_0) = c$.*

Proof. Without loss of generality assume that $x_1 < x_2$. Let

$$S = \{x \in [x_1, x_2] : \ f(x) < c\}.$$

Then S is non-empty (since $x_1 \in S$) and bounded above (since $x \le x_2$ for all $x \in S$). Let

$$\alpha := \sup S.$$

Then there exists a sequence (a_n) in S such that $a_n \to \alpha$. Note that $\alpha \in [x_1, x_2]$. Hence, by continuity of f, $f(a_n) \to f(\alpha)$. Since $f(a_n) < c$ for all $n \in \mathbb{N}$, we have $f(\alpha) \le c$. Note that $\alpha \ne x_2$, since $f(\alpha) \le c < f(x_2)$.

Now, let (b_n) be a sequence in (α, x_2) such that $b_n \to \alpha$. Then, again by continuity of f, $f(b_n) \to f(\alpha)$. Since $b_n > \alpha$, $b_n \notin S$ and hence $f(b_n) \ge c$. Therefore, $f(\alpha) \ge c$. Thus, we have prove that there exists $x_0 := \alpha$ such that $f(x_0) \le c \le f(x_0)$ so that $f(x_0) = c$. ∎

The following two corollaries are immediate consequences of the above theorem.

Theorem 2.26 *Let f be a continuous function defined on an interval. Then range of f is an interval.*

Corollary 2.27 *Suppose f is a continuous real valued function defined on an interval I. If $a, b \in I$ satisfy $a < b$ and $f(a)f(b) < 0$, then there exists $x_0 \in I$ such that*

$$a \le x_0 \le b \quad \textit{and} \quad f(x_0) = 0.$$

[1]Proof taken from the book: *A Course in Calculus and Real Analysus* by S.R. Ghorpade and B.V. Limaye (IIT Bombay), Springer, 2006.

2.3 Differentiability of functions

Definition 2.14 Suppose f is a (real valued) function defined on an open interval I and $x_0 \in I$. Then f is said to be **differentiable at** x_0 if

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists, and in that case the value of the limit is called the **derivative** of f at x_0.

The derivative of f at x_0, if exists, is denoted by

$$f'(x_0) \quad \text{or} \quad \frac{df}{dx}(x_0).$$

□

Exercise 2.13 Show that $f : I \to \mathbb{R}$ is differentiable at $x_0 \in I$ if and only if $\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}$ exists. ◀

Exercise 2.14 Suppose f is defined on an open interval I and $x_0 \in I$. Show that f is differentiable at $x_0 \in I$ if and only if there exists a continuous function $\Phi(x)$ such that

$$f(x) = f(x_0) + \Phi(x)(x - x_0),$$

and in that case $\Phi(x_0) = f'(x_0)$. ◀

Exercise 2.15 Let Φ be as in Exercise 2.14. Then f is differentiable at x_0, if and only if for every sequence (x_n) in $I \setminus \{x_0\}$ which converges to x_0, the sequence $\Phi(x_n)$ converges, and in that case $f'(x_0) = \lim_{n\to\infty} \Phi(x_n)$. ◀

Exercise 2.16 Suppose f and g defined on I are differentiable at $x_0 \in I$ and $\alpha \in \mathbb{R}$. Show that the functions $\varphi(x) := f(x) + g(x)$ and $\psi(x) := \alpha f(x)$, $x \in I$ are differentiable at x_0, and

$$\varphi'(x_0) = f'(x_0) + g'(x_0), \qquad \psi'(x_0) = \alpha\varphi'(x_0).$$

◀

2.3.1 Some properties of differentiable functions

Theorem 2.28 (Differentiability implies continuity) *Suppose f defined on I is differentiable at $x_0 \in I$. Then f is continuous at x_0.*

Proof. Note that

$$f(x_0 + h) - f(x_0) = \frac{f(x_0 + h) - f(x_0)}{h} h \to f'(x_0).0 = 0 \quad \text{as} \quad h \to 0.$$

Thus, f is continuous at x_0. ∎

For the following theorem, we may recall that if φ is continuous at a point x_0 and $\varphi(x_0) \neq 0$, then there exists an open interval I_0 containing x_0 such that $\varphi(x) \neq 0$ for all $x \in I_0$.

Theorem 2.29 (Products and quotient rules) *Suppose f and g defined on I are differentiable at $x_0 \in I$. Then the function $\varphi(x) := f(x)g(x)$ is differentiable at x_0, and*

$$\varphi'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0). \qquad (*)$$

If $g(x_0) \neq 0$, then the function $\psi(x) := f(x)/g(x)$ is differentiable at x_0, and

$$\psi'(x_0) = \frac{g(x_0)f'(x_0) - f(x_0)g'(x_0)}{[g(x_0)]^2}. \qquad (**)$$

Proof. Note that

$$\begin{aligned} \varphi(x_0+h) - \varphi(x_0) &= f(x_0+h)g(x_0+h) - f(x_0)g(x_0) \\ &= [f(x_0+h) - f(x_0)]g(x_0+h) + f(x_0)[g(x_0+h) - g(x_0)] \end{aligned}$$

so that

$$\begin{aligned} \frac{\varphi(x_0+h) - \varphi(x_0)}{h} &= \frac{f(x_0+h) - f(x_0)}{h} g(x_0+h) + f(x_0)\frac{g(x_0+h) - g(x_0)}{h} \\ &\to f'(x_0)g(x_0) + f(x_0)g'(x_0) \quad \text{as} \quad h \to 0. \end{aligned}$$

Hence, φ is differentiable at x_0, and

$$\varphi'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0).$$

Also, since

$$\begin{aligned} \psi(x_0+h) - \psi(x_0) &= \frac{f(x_0+h)g(x_0) - f(x_0)g(x_0+h)}{g(x_0+h)g(x_0)} \\ &= \frac{[f(x_0+h) - f(x_0)]g(x_0) - f(x_0)[g(x_0+h) - g(x_0)]}{g(x_0+h)g(x_0)}, \end{aligned}$$

we have

$$\begin{aligned} \frac{\psi(x_0+h) - \psi(x_0)}{h} &= \frac{\frac{f(x_0+h)-f(x_0)}{h} g(x_0) - f(x_0)\frac{g(x_0+h)-g(x_0)}{h}}{g(x_0+h)g(x_0)} \\ &\to \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{[g(x_0)]^2} \quad \text{as} \quad h \to 0. \end{aligned}$$

Thus, ψ is differentiable at x_0, and $\psi'(x_0) = \frac{g(x_0)f'(x_0)-f(x_0)g'(x_0)}{[g(x_0)]^2}$. ∎

Theorem 2.30 (Composition rule) *Suppose f is defined on an open interval I and $x_0 \in I$, and g is defined in an open interval containing $y_0 := f(x_0)$. Let*

$$\varphi = g \circ f.$$

Then we have the following.

(i) *Suppose f is differentiable at x_0 and g is differentiable at y_0. Then φ is differentiable at x_0 and*

$$\varphi'(x_0) = g'(y_0)f'(x_0).$$

(ii) *Suppose φ is differentiable at x_0, g is differentiable at y_0 and $g'(y_0) \neq 0$. Then f is differentiable at x_0 and*

$$f'(x_0) = \frac{\varphi'(x_0)}{g'(y_0)}.$$

(iii) *Suppose φ is differentiable at x_0, f is differentiable at x_0 and $f'(x_0) \neq 0$. Then g is differentiable at y_0 and*

$$g'(y_0) = \frac{\varphi'(x_0)}{f'(x_0)}.$$

Proof. For $x \neq x_0$, let

$$\Phi(x) := \frac{(g \circ f)(x) - (g \circ f)(x_0)}{x - x_0}.$$

Let (x_n) be a sequence in $I \setminus \{x_0\}$ which converges to x_0. Then, taking $y_n := f(x_n)$, $n \in \mathbb{N}$, and $y_0 = f(x_0)$, we have

$$\begin{aligned} \Phi(x_n) &= \frac{(g \circ f)(x_n) - (g \circ f)(x_0)}{x_n - x_0} \\ &= \frac{g(y_n) - g(y_0)}{x_n - x_0} \\ &= \frac{g(y_n) - g(y_0)}{y_n - y_0} \times \frac{f(x_n) - f(x_0)}{x_n - x_0}. \end{aligned}$$

(i) Suppose f is differentiable at x_0. Now, since $x_n \to x_0$ we have, by continuity of f at x_0, $y_n \to y_0$. Therefore,

$$\frac{f(x_n) - f(x_0)}{x_n - x_0} \to f'(x_0), \qquad \frac{g(y_n) - g(y_0)}{y_n - y_0} \to g'(y_0).$$

Thus, $\Phi(x_n) \to g'(y_0)f'(x_0)$ showing that $g \circ f$ is differentiable at x_0 and

$$(g \circ f)'(x_0) = g'(y_0)f'(x_0).$$

(ii) Suppose $g \circ f$ is is differentiable at x_0 and $g'(y_0) \neq 0$. Then we have $\Phi(x_n) \to (g \circ f)'(x_0)$ and

$$\frac{f(x_n) - f(x_0)}{x_n - x_0} = \frac{\Phi(x_n)}{\frac{g(y_n) - g(y_0)}{y_n - y_0}} \to \frac{(g \circ f)'(x_0)}{g'(y_0)}.$$

Hence, f is differentiable at x_0 and $f'(x_0) = \dfrac{(g \circ f)'(x_0)}{g'(y_0)}$.

(iii) Proof of this part is analogous to the proof of (ii). Hence, we omit the details. ∎

Remark 2.6 The part (ii) and (iii) of Theorem 2.30 is not available in standard books on Calculus. I found it useful while discussing derivative of logarithm function in next section (Section **??**). ♦

Exercise 2.17 Prove part (iii) of Theorem 2.30. ◄

We shall assume that the students are familiar with the following:

- For $c \in \mathbb{R}$, if $f(x) = c$, $x \in \mathbb{R}$, then $f'(x) = 0 \quad \forall\, x \in \mathbb{R}$.
- If $f(x) = x$, $x \in \mathbb{R}$, then $f'(x) = 1 \quad \forall\, x \in \mathbb{R}$.
- If $f(x) = \sin x$, $x \in \mathbb{R}$, then $f'(x) = \cos x \quad \forall\, x \in \mathbb{R}$.
- If $f(x) = \cos x$ then $f'(x) = -\sin x$.

From these, using theorems in the last subsection, we obtain the following:

- For $n \in \mathbb{N}$, if $f(x) = x^n$, $x \in \mathbb{R}$, then $f'(x) = nx^{n-1} \quad \forall\, x \in \mathbb{R}$.
- If $f(x) = \cos x = 1 - 2\sin^2(x/2)$, $x \in \mathbb{R}$, then $f'(x) = -\sin x \quad \forall x \in \mathbb{R}$.
- If $f(x) = \tan x$ for $x \in D := \{x \in \mathbb{R} : \cos x \neq 0\}$, then $f'(x) = sec^2 x \quad \forall\, x \in D$.

Example 2.24 The function e^x is differentiable for every $x \in \mathbb{R}$ and

$$(e^x)' = e^x \qquad \forall\, x \in \mathbb{R}.$$

We note that for $h \neq 0$,

$$\frac{e^{x+h} - e^x}{h} - e^x = \frac{e^x}{h}(e^h - 1 - h) = \frac{e^x}{h}\sum_{n=2}^{\infty}\frac{h^n}{n!}.$$

Now, if $|h| \leq 1$, then $|h|^n \leq |h|^2$ for all $n \in \{2, 3, \ldots\}$. Thus,

$$|h| \leq 1 \Longrightarrow \left|\frac{e^{x+h} - e^x}{h} - e^x\right| \leq e^x|h|\sum_{n=2}^{\infty}\frac{1}{n!} = e^x|h|(e - 2).$$

From this we obtain that e^x is differentiable at x and its derivative is e^x. □

Example 2.25 For $a > 0$, the function a^x is differentiable for every $x \in \mathbb{R}$ and

$$(a^x)' = a^x \ln a \qquad \forall\, x \in \mathbb{R}.$$

By the composition rule in Theorem 2.30,

$$(a^x)' = (e^{x \ln a})' = e^{x \ln a} \ln a = a^x \ln a.$$

□

Example 2.26 The function $\ln x$ is differentiable for every $x > 0$, and

$$(\ln x)' = \frac{1}{x}, \qquad x > 0.$$

To see this, let $f(x) = \ln x$ and $g(x) = e^x$. Then we have $g(f(x)) = x$ for every $x > 0$. Since $g \circ f$ is differentiable, g is differentiable, and $g'(y) = e^y \neq 0$ for every $y \in \mathbb{R}$, by Theorem 2.30, f is differentiable for every $x > 0$ and we have $g'(f(x))f'(x) = 1$. Thus,

$$1 = e^{\ln x}(\ln x)' = x(\ln x)'$$

so that $(\ln x)' = 1/x$. □

Example 2.27 For $a > 0$, the function $\log_a x$ is differentiable for every $x > 0$, and

$$(\log_a x)' = \frac{1}{x \ln a}, \qquad x > 0.$$

We know that

$$\log_a x = \frac{\ln x}{\ln a}.$$

Hence, $(\log_a x)' = \dfrac{1}{x \ln a}$ for every $x > 0$. □

Example 2.28 For $r \in \mathbb{R}$, let $f(x) = x^r$ for $x > 0$. Then f is differentiable for every $x > 0$ and

$$f'(x) = rx^{r-1}, \qquad x > 0.$$

By the composition rule in Theorem 2.30,

$$f'(x) = (e^{r \ln x})' = e^{r \ln x}\frac{r}{x} = \frac{x^r r}{x} = rx^{r-1}.$$

□

Exercise 2.18 Prove the following.

(i) The function $\ln |x|$ is differentiable for every $x \in \mathbb{R}$ with $x \neq 0$, and

$$(\ln |x|)' = \frac{1}{x}, \qquad x \neq 0.$$

(ii) For $a > 0$, the function $\log_a |x|$ is differentiable for every $x \in \mathbb{R}$ with $x \neq 0$, and

$$(\log_a |x|)' = \frac{1}{x \ln a}, \qquad x \neq 0.$$

◀

2.3.2 Maxima and minima

Recall from Theorem 2.24 that if $f : [a, b] \to \mathbb{R}$ is a continuous function, then there exists x_0, y_0 in $[a, b]$ such that

$$f(x_0) \le f(x) \le f(y_0) \qquad \forall x \in [a, b].$$

In this case, we write

$$f(x_0) = \min_{a \le x \le b} f(x) \quad \text{and} \quad f(y_0) = \max_{a \le x \le b} f(x).$$

Definition 2.15 A (real valued) function f defined on an interval I (of finite or infinite length) is said to attain

(a) **global maximum** at a point $x_1 \in I$ if $f(x_1) \ge f(x)$ for all $x \in I$, and

(b) **global minimum** at a point $x_2 \in I$ if $f(x_2) \le f(x)$ for all $x \in I$.

The function f is said to attain **global extremum** at a point $x_0 \in I$ if f attains either global maximum or global minimum at x_0. □

Thus, a continuous fiunction f defined on a closed and bounded interval I attain global maximum and global minimum at some points in I.

In Remark 2.5 we have seen that a function f defined on an interval I need not attain maximum or minimum if either I is not closed and bounded or if f is not continuous. However, maximum or minimum can attain in a subinterval. To take care of these cases, we introduce the following definition.

Definition 2.16 A (real valued) function f defined on an interval I (of finite or infinite length) is said to attain

(a) **local maximum** at a point $x_1 \in I$ if there exists $\delta > 0$ such that

$$f(x_1) \ge f(x) \qquad \forall\, x \in I \cap (x_1 - \delta, x_1 + \delta),$$

(b) **local minimum** at a point $x_2 \in$ if if there exists $\delta > 0$ such that

$$f(x_2) \le f(x) \qquad \forall\, x \in I \cap (x_2 - \delta, x_2 + \delta).$$

The function f is said to attain **local extremum** at a point $x_0 \in I$ if f attains either local maximum or local minimum at x_0. □

Remark 2.7 It is conventional to omit the adjective *local* in local maximum, local minimum and local extremum. Thus when we say a function has maximum at a point x_0, we generally mean a local maximum at x_0. Similar comments apply to minimum and extremum. ♦

Theorem 2.31 (Necessary condition) *Suppose f is a continuous function defined on an interval I having local extremum at a point $x_0 \in I$. If x_0 is an interior point of I (i.e., x_0 is not an end point of I) and f is differentiable at x_0, then $f'(x_0) = 0$.*

Proof. Suppose f attains local maximum at x_0 which is an interior point of I. Then there exists $\delta > 0$ such that $(x_0 - \delta, x_0 + \delta) \subseteq I$ and $f(x_0) \geq f(x_0 + h)$ for all h with $|h| < \delta$. Hence, for all h with $|h| < \delta$,

$$\frac{f(x_0 + h) - f(x_0)}{h} \geq 0 \quad \text{if} \quad h < 0,$$

$$\frac{f(x_0 + h) - f(x_0)}{h} \leq 0 \quad \text{if} \quad h > 0.$$

Taking limit as $h \to 0$, we get $f'(x_0) \geq 0$ and $f'(x_0) \leq 0$ so that $f'(x_0) = 0$.

By analogous arguments, it can be shown that if f attains minimum at a point $y_0 \in (a, b)$, then $f'(y_0) = 0$. ∎

Remark 2.8 A function can have more than one maximum and minimum. For example, consider

$$f(x) = \sin(4x), \qquad [0, \pi].$$

We see that f has maximum value 1 at $\pi/8$ and $5\pi/8$, and has minimum value -1 at $3\pi/8$ and $7\pi/8$. ♦

Remark 2.9 (a) In view of Theorem 2.31, if a function f is differentiable at an interior point x_0 of an interval I and $f'(x_0) \neq 0$, then f can not have local maximum or local minimum at x_0.

(b) It is to be observed that in order to have a maximum or minimum at a point x_0, the function need not be differentiable at x_0. For example

$$f(x) = 1 - |x|, \qquad |x| \leq 1,$$

has a maximum at 0 and

$$g(x) = |x|, \qquad |x| \leq 1,$$

has a minimum at 0. Both f and g are not differentiable at 0.

(c) Also, if a function is differentiable at a point x_0 and $f'(x_0) = 0$, then it is not necessary that it has loal maximum or local minimum at x_0. For example, consider

$$f(x) = x^3, \qquad |x| < 1.$$

In this example, we have $f'(0) = 0$. Note that f has neither local maximum nor local minimum at 0. ♦

Definition 2.17 Suppose f is defined on an interval I and x_0 is an interior point of I. If $f'(x_0)$ exists and $f'(x_0) = 0$ or if $f'(x_0)$ does not exist, then x_0 is called a **critical point** of f. □

In Section 2.3.5 we shall give some sufficient conditions for existence of local exrema of functions. Now, let us derive some important consequences of Theorem 2.31.

2.3.3 Some important theorems

Rolle's theorem

Theorem 2.32 (Rolle's theorem) *Suppose f is a continuous function defined on a closed and bounded interval $[a, b]$ such that it is differentiable at every $x \in (a, b)$. If $f(a) = f(b)$, then there exists $c \in (a, b)$ such that $f'(c) = 0$.*

Proof. Let $g(x) = f(x) - f(a)$. Then we have $g(a) = 0 = g(b)$, and $g'(x) = f'(x)$ for every $x \in (a, b)$.

We know that g attain maximum and minimum at some points x_1 and x_2, respectively, in $[a, b]$, i.e., there exists x_1, x_2 in $[a, b]$ such that

$$g(x_2) \leq g(x) \leq g(x_1) \qquad \forall\, x \in [a, b].$$

If $g(x_1) = g(x_2)$, then g is a constant function and hence $g'(x) = 0$ for all $x \in [a, b]$. Hence, assume that $g(x_2) < g(x_1)$. Then, either $g(x_1) \neq 0$ or $g(x_2) \neq 0$. Assume that $g(x_2) \neq 0$, so that $x_2 \neq a$ and $x_2 \neq b$. Hence, by Theorem 2.31, $g'(x_2) = 0$. Thus, $f'(x_2) = 0$.

Similarly, if $g(x_1) \neq 0$, then we shall arrive at $f'(x_1) = 0$. ∎

Exercise 2.19 Show that between any two roots of the equation $e^x \cos x - 1 = 0$, there is at least one root of the equation $e^x \sin x - 1 = 0$. ◀

Lagrange's mean value theorem

As a corollary to Rolle's theorem we obtain the following.

Theorem 2.33 (Lagrange's mean value theorem) *Suppose f is a continuous function defined on a closed and bounded interval $[a, b]$ such that it is differentiable at every $x \in (a, b)$. Then there exists $c \in (a, b)$ such that*

$$f(b) - f(a) = f'(c)(b - a).$$

Proof. Let

$$\varphi(x) := f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a), \qquad x \in [a, b].$$

Note that φ is continuous on $[a,b]$, differentiable in (a,b), $\varphi(a) = 0 = \varphi(b)$, and

$$\varphi'(x) := f'(x) - \frac{f(b) - f(a)}{b-a}, \qquad x \in (a,b).$$

By Rolle's theorem (Theorem 2.32), there exists $c \in (a,b)$ such that $\varphi'(c) = 0$. Thus, $f(b) - f(a) = f'(c)(b-a)$. ∎

Example 2.29 Let f be continuous on $[a,b]$ and differentiable at every point in (a,b). Suppose there exists $c \in \mathbb{R}$ such that

$$f'(x) = c \qquad x \in (a,b).$$

Then there exists $b \in \mathbb{R}$ such that

$$f(x) = c\,x + b \qquad \forall\, x \in [a,b].$$

In particular, $f'(x) = 0$ for all $x \in (a,b)$, then f is a constant function.

To see this consider $x_0 \in (a,b)$. Then for any $x \in [a,b]$, there exists ξ_x between x_0 and x such that

$$f(x) - f(x_0) = f'(\xi_x)(x - x_0) = c(x - x_0).$$

Hence, $f(x) = f(x_0) + c(x - x_0)$. Thus, $f(x) = c\,x + b$ with $b = f(x_0) - c\,x_0$. □

Cauchy's generalized mean value theorem

Suppose f and g are continuous functions on $[a,b]$ which are differentiable on (a,b). Suppose further that $g'(x) \neq 0$ for all $x \in (a,b)$. Then, by Lagrange's mean value theorem, there exist c_1, c_2 in (a,b) such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c_1)}{g'(c_2)}.$$

Question is whether we can assert the existence of a single point $c \in (a,b)$ such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

Answer is in affirmative as the following theorem shows.

Theorem 2.34 (Cauchy's generalized mean value theorem) *Suppose f and g are continuous functions on $[a,b]$ which are differentiable at every point in (a,b). Suppose further that $g'(x) \neq 0$ for all $x \in (a,b)$. Then, there exists $c \in (a,b)$ such that*

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

Proof. First note that from the assumption on g, using Mean value theorem, $g(b) \neq g(a)$. Now, let

$$\varphi(x) := f(x) - f(a) - \frac{f(b) - f(a)}{g(b) - g(a)}[g(x) - g(a)], \qquad x \in [a, b].$$

Note that φ is continuous on $[a, b]$, differentiable in (a, b), $\varphi(a) = 0 = \varphi(b)$, and

$$\varphi'(x) := f'(x) - \frac{f(b) - f(a)}{g(b) - g(a)} g'(x), \qquad x \in (a, b).$$

By Rolle's theorem (Theorem 2.32), there exists $c \in (a, b)$ such that $\varphi'(c) = 0$. This completes the proof. ▮

Exercise 2.20 Let $0 < a < b$. Show that for every $n \in \mathbb{N}$, $a < \dfrac{n[b^{n+1} - a^{n+1}]}{(n+1)[b^n - a^n]} < b$.

[Hint: take $f(x) = x^{n+1}$ and $g(x) = x^n$.] ◀

If f is defined in a closed interval $[a, b]$ and $x_0 = a$ or $x_0 = b$, then by $lim_{x \to x_0} f(x)$ we mean $lim_{x \to x_0^+} f(x)$ if $x_0 = a$ and $lim_{x \to x_0^-} f(x)$ if $x_0 = b$.

L'Hospital's rules

Theorem 2.35 (L'Hospital's rule)[2] *Suppose f and g are continuous functions on $[a, b]$ which are differentiable at every point in (a, b), except possibly at $x_0 \in [a, b]$. Suppose $f(x_0) = 0 = g(x_0)$ and $\lim_{x \to x_0} \frac{f'(x)}{g'(x)}$ exists. Then $\lim_{x \to x_0} \frac{f(x)}{g(x)}$ exists and*

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}.$$

Proof. Since $\lim_{x \to x_0} \frac{f'(x)}{g'(x)}$ exists, there exists a deleted neighbourhood D_0 of x_0 such that $g'(x) \neq 0$ for $x \in D_0 \cap [a, b]$. By Cauchy's generalized mean value theorem (Theorem 2.34), for every $x \in D_0$, there exists ξ_x between x and x_0 such that

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(x_0)}{g(x) - g(x_0)} = \frac{f'(\xi_x)}{g'(\xi_x)}.$$

Since $|\xi_x - x_0| < |x - x_0|$ and $\lim_{x \to x_0} \frac{f'(x)}{g'(x)}$ exists, by using the limits of composition of functions, $\lim_{x \to x_0} \frac{f'(\xi_x)}{g'(\xi_x)}$ exists and it is equal to $\lim_{x \to x_0} \frac{f'(x)}{g'(x)}$ (See Exercise 2.5). Thus, $\lim_{x \to x_0} \frac{f(x)}{g(x)}$ exists and $\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}$. This completes the proof. ▮

[2] *L'Hospital* is pronounced as *Lopital.* The rule is named after the 17th-century French mathematician Guillaume de l'Hospital, who published the rule in his book *Analyse des Infiniment Petits pour l'Intelligence des Lignes Courbes* (i.e., Analysis of the Infinitely Small to Understand Curved Lines) (1696), the first textbook on differential calculus.

The following theorem is proved by modifying the arguments in the proof of Theorem 2.38 .

Theorem 2.36 (L'Hospital's rule) *Suppose f and g are continuous functions on $[a,b]$ which are differentiable at every point in (a,b), except possibly at $x_0 \in [a,b]$. Suppose $\lim_{x\to x_0} f(x) = 0 = \lim_{x\to x_0} g(x)$ and $\lim_{x\to x_0} \frac{f'(x)}{g'(x)}$ exists. Then $\lim_{x\to x_0} \frac{f(x)}{g(x)}$ exists and*

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f'(x)}{g'(x)}.$$

Proof. Let $\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \neq x_0 \\ 0 & \text{if } x = x_0 \end{cases}$ and $\tilde{g}(x) = \begin{cases} g(x) & \text{if } x \neq x_0 \\ 0 & \text{if } x = x_0 \end{cases}$. Then, the result is obtained from Theorem 2.38 by taking $\tilde{f}$ and $\tilde{g}$ in place of f and g, respectively. ∎

Theorem 2.37 (L'Hospital's rule) *Suppose f and g are differentiable at every point in (a,∞) for some $a > 0$. Suppose $\lim_{x\to\infty} f(x) = 0 = \lim_{x\to\infty} g(x)$ and $\lim_{x\to\infty} \frac{f'(x)}{g'(x)}$ exists. Then $\lim_{x\to\infty} \frac{f(x)}{g(x)}$ exists and*

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}.$$

Proof. Let $\tilde{f}(y) = f(1/y)$ and $\tilde{g}(y) = g(1/y)$ for $0 < y < 1/a$. We note that

$$\lim_{x\to\infty} f(x) = 0 = \lim_{x\to\infty} g(x) \iff \lim_{y\to 0} \tilde{f}(y) = 0 = \lim_{y\to 0} \tilde{g}(y).$$

Also, since

$$\tilde{f}'(y) = [f(1/y)]' = f'(1/y)(-1/y^2), \qquad \tilde{g}'(y) = [g(1/y)]' = g'(1/y)(-1/y^2),$$

we have

$$\lim_{x\to\infty} \frac{f'(x)}{g'(x)} \quad \text{exists} \quad \iff \lim_{y\to 0} \frac{\tilde{f}'(y)}{\tilde{g}'(y)} \quad \text{exists.}$$

Hence, applying Theorem 2.38 to $\tilde{f}$, $\tilde{g}$ instead of f, g, we obtain the result. ∎

The following theorem also holds.

Theorem 2.38 (L'Hospital's rule) *Suppose f and g are continuous functions on $[a,b]$ which are differentiable at every point in (a,b), except possibly at $x_0 \in [a,b]$. Suppose $\lim_{x\to x_0} f(x) = \infty = \lim_{x\to x_0} g(x)$ and $\lim_{x\to x_0} \frac{f'(x)}{g'(x)}$ exists. Then $\lim_{x\to x_0} \frac{f(x)}{g(x)}$ exists and $\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f'(x)}{g'(x)}$.*

Proof. Let $\beta := \lim_{x\to x_0} \dfrac{f'(x)}{g'(x)}$. First we consider the case of $\beta \neq 0$. In this case, since

$$\lim_{x\to x_0} f(x) = \infty = \lim_{x\to\infty} g(x) \iff \lim_{x\to x_0} (1/f(x)) = 0 = \lim_{x\to x_0} (1/g(x)),$$

the result follows from Theorem 2.37 by interchanging the roles of f and g.

To consider the general case where β is not necessarily non-zero, let $\alpha, x \in \mathbb{R}$ be such that $|x - x_0| < |\alpha - x_0|$ and α sufficiently close to x_0 such that $g(x) \neq g(\alpha)$. This is guaranteed because, $g'(x) \neq 0$ for x sufficiently close to x_0. Then there exists $\xi_{x,\alpha}$ lying between α and x such that have

$$\frac{f'(\xi_{x,\alpha})}{g'(\xi_{x,\alpha})} = \frac{f(x) - f(\alpha)}{g(x) - g(\alpha)} = \frac{f(x)}{g(x)} \frac{\left[1 - \frac{f(\alpha)}{f(x)}\right]}{\left[1 - \frac{g(\alpha)}{g(x)}\right]}$$

so that

$$\frac{f(x)}{g(x)} = \frac{f'(\xi_{x,\alpha})}{g'(\xi_{x,\alpha})} \frac{\left[1 - \frac{f(\alpha)}{f(x)}\right]}{\left[1 - \frac{g(\alpha)}{g(x)}\right]}.$$

Now, for each such α we have

$$\lim_{x\to x_0} \left[1 - \frac{f(\alpha)}{f(x)}\right] = 1 = \lim_{x\to x_0} \left[1 - \frac{g(\alpha)}{g(x)}\right].$$

Also, since $|\xi_{x,\alpha} - x_0| < |\alpha - x_0|$, by using the limits of composition of functions (See Exercise 2.5) we have

$$\lim_{\alpha\to x_0} \frac{f'(\xi_{x,\alpha})}{g'(\xi_{x,\alpha})} = \lim_{x\to x_0} \frac{f'(x)}{g'(x)}.$$

Hence, using (ε, δ) arguments, it can be shown that $\lim_{\alpha\to x_0} \dfrac{f'(\xi_{x,\alpha})}{g'(\xi_{x,\alpha})} \dfrac{\left[1 - \frac{f(\alpha)}{f(x)}\right]}{\left[1 - \frac{g(\alpha)}{g(x)}\right]}$ exists so that $\lim_{x\to x_0} \dfrac{f(x)}{g(x)}$ exists and

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{\alpha\to x_0} \frac{f'(\xi_{x,\alpha})}{g'(\xi_{x,\alpha})} \frac{\left[1 - \frac{f(\alpha)}{f(x)}\right]}{\left[1 - \frac{g(\alpha)}{g(x)}\right]} = \lim_{x\to x_0} \frac{f'(x)}{g'(x)}.$$

This completes the proof. ∎

Exercise 2.21 Fill gaps in the proof of the above theorem. ◄

Remark 2.10 The cases

(i) $\lim_{x\to -\infty} f(x) = 0 = \lim_{x\to -\infty} g(x)$,

(ii) $\lim_{x\to x_0} f(x) = -\infty = \lim_{x\to x_0} g(x)$

can be treated analogously to the cases already discussed in the above theorems. ♦

Taylor's formula

Now, we state another important formula in calculus.

Theorem 2.39 (Taylor's formula[3]) *Suppose f is defined and has derivatives $f^{(1)}(x), f^{(2)}(x), \ldots, f^{(n+1)}(x)$ for x in an open interval I and $x_0 \in I$. Then, for every $x \in I$, there exists ξ_x between x and x_0 such that*

$$f(x) = f(x_0) + \sum_{j=1}^{n} \frac{f^{(j)}(x_0)}{j!}(x - x_0)^j + \frac{f^{(n+1)}(\xi_x)}{(n+1)!}(x - x_0)^{n+1}.$$

Equivalently, for h with $x_0 + h \in I$, there exists ξ_x between x and x_0 such that

$$f(x_0 + h) = f(x_0) + \sum_{j=1}^{n} \frac{f^{(j)}(x_0)}{j!}h^j + \frac{f^{(n+1)}(\xi_x)}{(n+1)!}h^{n+1}.$$

Proof. [4] Let $x \in I$ with $x \neq x_0$, and let

$$P_n(t) = f(x_0) + \sum_{j=1}^{n} \frac{f^{(j)}(x_0)}{j!}(t - x_0)^j, \qquad t \in I.$$

Then P_n is a polynomial of degree n, $P_n(x_0) = f(x_0)$ and

$$P_n^{(j)}(x_0) = f^{(j)}(x_0), \qquad j \in \{1, \ldots, n\}.$$

Now, let

$$g(t) = f(t) - P_n(t) - \varphi(x)(t - x_0)^{n+1}, \qquad t \in I,$$

where

$$\varphi(x) := \frac{f(x) - P_n(x)}{(x - x_0)^{n+1}}.$$

Note that, by this choice of $\varphi(x)$, we have $g(x_0) = 0$ and $g(x) = 0$. Also, we have

$$g^{(1)}(x_0) = 0, \quad g^{(2)}(x_0) = 0, \quad \ldots, \quad g^{(n)}(x_0) = 0.$$

Hence, by Rolle's theorem, there exists x_1 between x_0 and x such that $g'(x_1) = 0$. Again, by Rolle's theorem, there exists x_2 between x_0 and x_1 such that $g''(x_2) = 0$. Continuing this, there exists $\xi_x := x_{n+1}$ between x_0 and x_n such that $g^{(n+1)}(\xi_x) = 0$. But,

$$g^{(n+1)}(t) = f^{(n+1)}(t) - P_n^{(n+1)}(t) - \varphi(x)(n+1)! = f^{(n+1)}(t) - \varphi(x)(n+1)!.$$

Thus, we have

$$\varphi(x) = \frac{f^{(n+1)}(\xi_x)}{(n+1)!},$$

---

[3]In the next Chapter we shall give another, rather *simpler* proof for this.

[4]This proof adapted from S. Ghorpade & B.V. Limaye: A Course in Calculus and Analysis, Springer, 2006

so that

$$f(x) = f(x_0) + \sum_{j=1}^{n} \frac{f^{(j)}(x)}{j!}(x - x_0)^j + \frac{f^{(n+1)}(\xi_x)}{(n+1)!}(x - x_0)^{n+1}.$$

Thus the proof is complete. ▮

Definition 2.18 In the Taylor's formula (Theorem 2.39), the term

$$R_n(x) := \frac{f^{(n+1)}(\xi_x)}{(n+1)!}(x - x_0)^{n+1}$$

is called the **remainder term** in the formula. □

We observe that if f is infinitely differentiable and if

$$|R_n(x)| \to 0 \quad \text{as} \quad n \to \infty$$

for every $x \in I$, then

$$f(x) = f(x_0) + \sum_{n=1}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n, \qquad x \in I. \tag{$*$}$$

Definition 2.19 If f can be represented as a series as in ($*$), for all x in a neighbourhood of x_0, then such a series is called the **Taylor's series** of f around the point x_0. □

A natural question that one may ask is:

> Does every infinitely differentiable function in a neighbourhood of x_0 has a Taylor's series expansion?

Unfortunately, the answer is negative. For example, if we define

$$f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0, \\ 0, & x = 0, \end{cases}$$

then it can be seen that $f(0) = 0$ and $f^{(k)}(0) = 0$ for all $k \in \mathbb{N}$. Thus, f does not have the Taylor's series expansion around the point 0.

Example 2.30 Using Taylor's formula, we shall show that

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \qquad \forall\, x \in \mathbb{R}.$$

For this, let $f(x) = \sin x$ and $x_0 = 0$. Since f is infinitely differentiable, and

$$f^{2j}(0) = 0, \qquad f^{2j-1}(0) = (-1)^j \qquad \forall\, j \in \mathbb{N},$$

we have

$$\begin{aligned} f(x) &= f(x_0) + \sum_{j=1}^{2n+1} \frac{f^{(j)}(0)}{j!} x^j + \frac{f^{(2n+2)}(\xi_x)}{(2n+2)!} x^{2n+2} \\ &= f(x_0) + \sum_{j=0}^{n} \frac{f^{(2j+1)}(0)}{(2j+1)!} x^{2j+1} + \frac{f^{(2n+2)}(\xi_x)}{(2n+2)!} x^{2n+2} \\ &= f(x_0) + \sum_{j=0}^{n} \frac{(-1)^j}{(2j+1)!} x^{2j+1} + \frac{f^{(2n+2)}(\xi_x)}{(2n+2)!} x^{2n+2} \end{aligned}$$

Also, since $|\sin x| \le 1$, we have

$$\left| \frac{f^{(2n+2)}(\xi_x) x^{2n+2}}{(2n+2)!} \right| \le \frac{|x|^{2n+2}}{(2n+2)!} \to 0 \quad \text{as} \quad n \to \infty.$$

Therefore,

$$\left| f(x) - \left[f(x_0) + \sum_{j=0}^{n} \frac{(-1)^j}{(2j+1)!} x^{2j+1} \right] \right| \to 0 \quad \text{as} \quad n \to \infty$$

and hence, $\sin x = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \qquad \forall\, x \in \mathbb{R}.$ □

Exercise 2.22 Suppose f is infinitely differentiable in an open interval I and $x_0 \in I$. Further, suppose that there exists $M > 0$ such that

$$|f^{(k)}(x)| \le M \qquad \forall x \in I, \quad \forall k \in \mathbb{N} \cup \{0\}.$$

Then show that

$$f(x) = f(x_0) + \sum_{n=1}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n, \qquad x \in I.$$

◀

Exercise 2.23 Using Taylor's formula, prove the following:

(i) $\cos x = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$ for all $x \in \mathbb{R}$.

(ii) $e^x = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all $x \in \mathbb{R}$.

(iii) $\displaystyle\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for all x with $|x| < 1$.

(iv) $\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$ for all $x \in \mathbb{R}$.

Deduce Madhava-Gregory series: $\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$.

◀

2.3.4 Increasing and decreasing functions

Definition 2.20 Let f be a function on a set $D \subseteq \mathbb{R}$. Then f is said to be

(i) **monotonically increasing** or **increasing** on D if

$$x, y \in D, \quad x \le y \Longrightarrow f(x) \le f(y),$$

(ii) **strictly increasing** on D if

$$x, y \in D, \quad x < y \Longrightarrow f(x) < f(y),$$

(iii) **monotonically decreasing** or **decreasing** on D if

$$x, y \in D, \quad x \le y \Longrightarrow f(x) \ge f(y).$$

(iv) **strictly decreasing** on D if

$$x, y \in D, \quad x < y \Longrightarrow f(x) > f(y).$$

□

Theorem 2.40 *Let f be continuous on $[a, b]$ and differentiable on (a, b). Then*

(i) *f is increasing iff $f'(x) \ge 0$ for all $x \in (a, b)$.*

(ii) *f is decreasing iff $f'(x) \le 0$ for all $x \in (a, b)$.*

(iii) *f is strictly increasing if $f'(x) > 0$ for all $x \in (a, b)$.*

(iv) *f is strictly decreasing if $f'(x) < 0$ for all $x \in (a, b)$.*

Proof. (i) Suppose f is increasing and $x \in (a, b)$. Note that

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0^+} \frac{f(x+h) - f(x)}{h}.$$

Since, $\frac{f(x+h)-f(x)}{h} \ge 0$ for $h > 0$, from the above equality we obtain $f'(x) \ge 0$.

To see the converse and (iii), let $x_1, x_2 \in [a, b]$ with $x_1 < x_2$. Then, by mean value theorem, there exists $\xi \in (x_1, x_2)$ such that

$$f(x_2) - f(x_1) = f'(\xi)(x_2 - x_1).$$

Hence, if $f'(x) \geq 0$ (respectively, $f'(x) > 0$) for every $x \in (a, b)$, then $f(x_1) \leq f(x_2)$ (respectively, $f(x_1) < f(x_2)$). Thus, (i) and (iii) are proved. Similar arguments will lead to the proof of (ii) and (iv). ∎

Example 2.31 Consider the function $f(x) = x^4$ for $x \in \mathbb{R}$. Then we have $f'(x) = 4x^3$ for all $x \in \mathbb{R}$. Note that

$$f'(x) > 0 \quad \forall\, x > 0 \quad \text{and} \quad f'(x) < 0 \quad \forall\, x < 0.$$

Hence,

f is strictly increasing on $(0, \infty)$, and

f is strictly decreasing on $(-\infty, 0)$. □

2.3.5 More about local maxima and local minima

Theorem 2.41 (A sufficient condition) *Suppose f is continuous on an interval I and x_0 is an interior point of I. Further suppose that f is differentiable in a deleted nbd of x_0.*

(i) *If there exists an open interval $I_0 \subseteq I$ containing x_0 such that*

$$f'(x) > 0 \quad \forall\, x \in I_0,\, x < x_0 \quad \textit{and} \quad f'(x) < 0 \qquad \forall\, x \in I_0,\, x > x_0,$$

then f has local maximum at x_0.

(ii) *If there exists an open interval $I_0 \subseteq I$ containing x_0 such that*

$$f'(x) < 0 \quad \forall\, x \in I_0,\, x < x_0 \quad \textit{and} \quad f'(x) > 0 \quad \forall\, x \in I_0,\, x > x_0,$$

then f has local minimum at x_0.

Proof. (i) Let $x \in I_0$. Then, by mean value theorem, there exists ξ_x between x_0 and x such that

$$f(x) - f(x_0) = f'(\xi_x)(x - x_0).$$

By assumption,

$$x < x_0 \Longrightarrow f'(\xi_x) > 0 \quad \text{and} \quad x > x_0 \Longrightarrow f'(\xi_x) < 0.$$

Hence, in both the cases, we have $f(x) < f(x_0)$ so that f has local maximum at x_0. Thus, (i) is proved.

Similar arguments will lead to the proof of (ii). ∎

Example 2.32 Consider

$$f(x) = x^4, \qquad g(x) = 1 - x^4, \quad |x| < 1.$$

Then $f'(x) = 4x^3$ is negative for $x < 0$ and positive for $x > 0$. Hence, by Theorem 2.41, f has local minimum at 0. Also, $g'(x) = -4x^3$ is positive for $x < 0$ and negative for $x > 0$. Hence, by Theorem 2.41, g has local maximum at 0. □

Remark 2.11 The conditions given in Theorem 2.41 cannot be dropped. For example, consider $f(x) = x^3$, $x \in \mathbb{R}$. Then $f'(x) = 3x^2 > 0$ for all $x \neq 0$. Note that f does not have extremum at 0. ♦

Theorem 2.42 (Another sufficient condition) *Suppose f is defined on an interval I and x_0 is an interior point of I. Further, suppose that f continuously twice differentiable in a neighbourhood of x_0 and $f'(x_0) = 0$. Then we have the following:*

(i) If $f''(x_0) < 0$, then f has local maximum at x_0.

(ii) If $f''(x_0) > 0$, then f has local minimum at x_0.

Proof. By Taylor's theorem, there exists an open interval I_0 containing x_0 such that for every $x \in I_0$, there exists ξ_x between x_0 and x such that

$$f(x) - f(x_0) = f'(x_0)(x - x_0) + \frac{f''(\xi_x)}{2}(x - x_0)^2 = \frac{f''(\xi_x)}{2}(x - x_0)^2. \qquad (*)$$

(i) Suppose $f''(x_0) < 0$. Since f'' is continuous in a nbd of x_0, there exists an open interval I_1 containing x_0 such that for all $x \in I_1$,

$$f''(x) \le \frac{f''(x_0)}{2}.$$

In particular, from $(*)$, we obtain

$$f(x) - f(x_0) = \frac{f''(\xi_x)}{2}(x - x_0)^2 < 0 \qquad \forall x \in I_1.$$

Thus, f has a maximum at x_0.

(ii) Suppose $f''(x_0) > 0$. Then, we obtain reverse of the inequalities in the proof of (i), and arrive the conclusion that f has a minimum at x_0. ∎

Remark 2.12 The conditions given in Theorem 2.42 are only sufficient conditions. There are functions f for which none of the conditions (i) and (ii) are satisfied at a point x_0, still f can have local extremum at x_0. For example, consider

$$f(x) = x^4, \qquad g(x) = 1 - x^4, \quad |x| < 1.$$

Then $f'(0) = 0 = g'(0)$, f has local minimum at 0 and g has local maximum at 0. But, $f''(0) = 0 = g''(0)$. ♦

Remark 2.13 How to identify critical points and extreme points of a function?

1. Suppose f is defined on an open interval I.

   (a) Find those points at which either f is not differentiable or f' vanish. These points are the critical points of f.

   (b) Suppose $f'(x_0) = 0$.

      i. If $f'(x)$ has the same sign for x on both side of x_0, then f does not have an extremum at x_0. Otherwise,

      ii. use the test for maximum or minimum as given in Theorem 2.41.

2. Suppose f is continuous on $[a, b]$ and differentiable on (a, b).

   (a) f can have maximum or minimum only the at the end points of $[a, b]$ or at those points in (a, b) at which f' vanishes.

   (b) Use the tests as in Theorem 2.41 or Theorem 2.42.

♦

2.4 Additional exercises

2.4.1 Limit

1. Using the definition of limit, show that $\lim_{x\to 3} \dfrac{x}{4x-9} = 1$.

2. Show that the function f defined by $f(x) = \begin{cases} x, & \text{if } x < 1, \\ 1 + x, & \text{if } x \geq 1 \end{cases}$ does not have the limit as $x \to 1$.

3. Let f be defined by $f(x) = \begin{cases} 3 - x, & \text{if } x > 1, \\ 1, & \text{if } x = 1, \\ 2x, & \text{if } x < 1. \end{cases}$

   Find $\lim_{x\to 1} f(x)$. Is it $f(1)$?

4. Let f be defined on a deleted neighbourhood D_0 of a point x_0 and $\lim_{x\to x_0} f(x) = b$. If $b \neq 0$, then show that there exists $\delta > 0$ such that $f(x) \neq 0$ for every $x \in (x_0 - \delta, x_0 + \delta) \cap D_0$.

5. Let f be defined by $f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q}, \\ 0, & \text{if } x \notin \mathbb{Q}. \end{cases}$ Show that

   (i) $\lim_{x\to 0} f(x)$ does not exist, and

   (ii) $\lim_{x\to 0} x f(x) = 0$.

6. Suppose $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to\infty} g(x) = b$. Show that $\lim_{x\to\infty} g(f(x)) = b$.

7. Let $f : (0,\infty) \to \mathbb{R}$ be such that $\lim_{x\to 0} f(x) = b$. Show that $\lim_{x\to\infty} f(x^{-1}) = b$.

2.4.2 Continuity

1. Suppose $f : [a,b] \to \mathbb{R}$ is continuous. If $c \in (a,b)$ is such that $f(c) > 0$, and if $0 < \beta < f(c)$, then show that there exists $\delta > 0$ such that $f(x) > \beta$ for all $x \in (c-\delta, c+\delta) \cap [a,b]$.

2. Let $f : \mathbb{R} \to \mathbb{R}$ satisfy the relation $f(x+y) = f(x) + f(y)$ for every $x, y \in \mathbb{R}$. If f is continuous at 0, then show that f is continuous at every $x \in \mathbb{R}$, and in that case $f(x) = xf(1)$ for every $x \in \mathbb{R}$.

3. There does not exist a continuous function f from $[0,1]$ onto $\mathbb{R}$ – Why?

4. Find a continuous function f from $(0,1)$ onto $\mathbb{R}$.

5. Suppose $f : [a,b] \to [a,b]$ is continuous. Show that there exists $c \in [a,b]$ such that $f(c) = c$.

6. There exists $x \in \mathbb{R}$ such that $17x^{19} - 19x^{17} - 1 = 0$ – Why?

7. If $p(x)$ is a polynomial of odd degree, then there exists at least one $\xi \in \mathbb{R}$ such that $p(\xi) = 0$.

8. Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous such that $f(x) \to 0$ as $|x| \to \infty$. Prove that f attains either a maximum or a minimum.

9. Suppose $f : [a,b] \to \mathbb{R}$ is continuous such that for every $x \in [a,b]$, there exists a $y \in [a,b]$ such that $|f(y)| \le \frac{|f(x)|}{2}$. Show that there exists $\xi \in [a,b]$ such that $f(\xi) = 0$.

10. Suppose $f : [a,b] \to [a,b]$ is continuous such that there $|f(x) - f(y)| \le \frac{1}{2}|x-y|$ for all $x, y \in [a,b]$. Show that there exists $\xi \in [a,b]$ such that $f(\xi) = \xi$.

2.4.3 Differentiation

1. Prove that the function $f(x) = |x|$, $x \in \mathbb{R}$ is not differentiable at 0.

2. Consider a polynomial $p(x) = a_0 + a_1x^2 + \ldots + a_nx^n$ with real coefficients $a_0, a_1, \ldots, a_n$ such that $a_0 + \frac{a_1}{2} + \frac{a_2}{3} + \ldots + \frac{a_n}{n+1} = 0$. Show that there exists $x_0 \in \mathbb{R}$ such that $p(x_0) = 0$.

   [Note that the conclusion need not hold if the condition imposed on the coefficients is dropped. To see this, consider $p(x) = 1 + x^2$.]

3. Let I and J be open intervals and $f : I \to J$ be bijective and differentiable at every $x_0 \in I$. If $f'(x_0) \neq 0$, then show that the inverse function $f^{-1} : J \to I$ is also differentiable at x_0 and and $(f^{-1})'(x_0) = 1/f'(x_0)$.

4. Using Taylor's theorem, show that

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots + x^n.$$

5. Show that there does not exist a function $f : [0,1] \to \mathbb{R}$ which is differentiable on $(0,1)$ such that $f'(x) = \begin{cases} 0, & \text{if } 0 < x < 1/2, \\ 1, & \text{if } 1/2 \leq x < 1. \end{cases}$

   [Hint: Use Example 2.29 in the interval $[0,1/2]$ and $[1/2,1]$ taking $x_0 = 1/2$, and show that the resulting function f is not differentiable at $x_0 = 1/2$.]

6. Suppose f is differentiable on $(0,\infty)$ and $\lim_{x\to\infty} f'(x) = 0$.

   Prove that $\lim_{x\to\infty} [f(x+1) - f(x)] = 0$.