2015

BOOKLET No. TEST CODE : PMB

Afternoon

Duration of test : 2 hours

Write your Registration number, Test Code, Number of this booklet, etc. in the appropriate places on your ANSWER BOOKLET.

This test has questions arranged in two groups.

Each group consists of 6 questions.

You need to answer 4 questions FROM EACH GROUP.

Each question carries 10 marks. Total marks = 80.

ALL ROUGH WORK MUST BE DONE ON THIS BOOKLET
AND/OR ON YOUR ANSWER BOOKLET.
CALCULATORS ARE NOT ALLOWED.

STOP ! WAIT FOR THE SIGNAL TO START

INSTRUCTIONS FOR CANDIDATES

- Please answer FOUR questions from EACH group.
- Each question carries 10 marks. Total marks : 80.
- $\mathbb{R}$ and $\mathbb{Q}$ denote the set of real numbers and the set of rational numbers respectively.

Group A

1. If (a_n) is a sequence in $(0,1)$, show that $\frac{1}{n}\sum_{k=1}^{n} a_k \to 0$ if and only if $\frac{1}{n}\sum_{k=1}^{n} a_k^2 \to 0$.
2. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function. Suppose $\delta = \inf_{x \in \mathbb{R}} f'(x) > 0$. Prove that $f(a) = 0$ for some $a \in \mathbb{R}$.
3. Show that $\int_0^\infty \left|\frac{\sin x}{x}\right| dx = \infty$.
4. Prove that a sequence of continuous functions $f_n : [0,1] \to \mathbb{R}$ converges uniformly to a continuous function f if and only if $f_n(x_n) \to f(x)$ whenever $x_n \to x$.
5. Let (X, d) be a metric space. If there exists an uncountable set $A \subset X$ such that $\inf\{d(x,y) : x \in A, y \in A, x \neq y\} > 0$, show that X is not separable. [This means there is no countable subset of X which is dense.] Hence or otherwise show the following metric space is not separable: X is the space of all functions $f : \mathbb{R} \to [0,1]$ and $d(f,g) = \sup\{|f(x) - g(x)| : x \in \mathbb{R}\}$.
6. Let y_1 and y_2 be solutions of $y'' + P(x)y' + Q(x)y = R(x)$ on $[a,b]$ where P, Q, R are continuous functions on $[a,b]$. Prove that either $y_1 = y_2$ or $\{x \in [a,b] \mid y_1(x) = y_2(x)\}$ is finite.

Group B

1. Let $T : X \to Y$ be an $\mathbb{R}$-linear map of $\mathbb{R}$-vector spaces X and Y of finite dimension. Let $W \subseteq Y$ be an $\mathbb{R}$ vector subspace such that $\operatorname{Im} T$ and W together span Y. Let $Z = T^{-1}(W)$. Show that

$$\dim X + \dim W = \dim Y + \dim Z.$$

2. Let $f \in \mathbb{Q}[X]$ be a polynomial of degree $n > 0$. Let $p_1, p_2, \ldots, p_{n+1}$ be distinct prime numbers. Show that there exists a non-zero polynomial $g \in \mathbb{Q}[X]$ such that $fg = \sum_{i=1}^{n+1} c_i X^{p_i}$ with $c_i \in \mathbb{Q}$.

3. Prove that the largest possible number of 1's in an $n \times n$ invertible matrix with all entries 0 or 1, is $n^2 - n + 1$.

4. Let G be a group which has only finitely many subgroups. Prove that G must be finite.

5. Let A be a commutative ring with unity. Prove that the set

$$Z = \{a \in A : ab = 0 \text{ for some nonzero } b \in A\}$$

contains a prime ideal of A.

6. Suppose the polynomial $X^4 + X + 1$ has multiple roots over a field of characteristic p. Determine the possible values of p.