Convergence Relations

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Relations

Theorem: Each arrow in Figure **??** indicates a direction of implication, e.g., almost sure convergence implies convergence in probability.

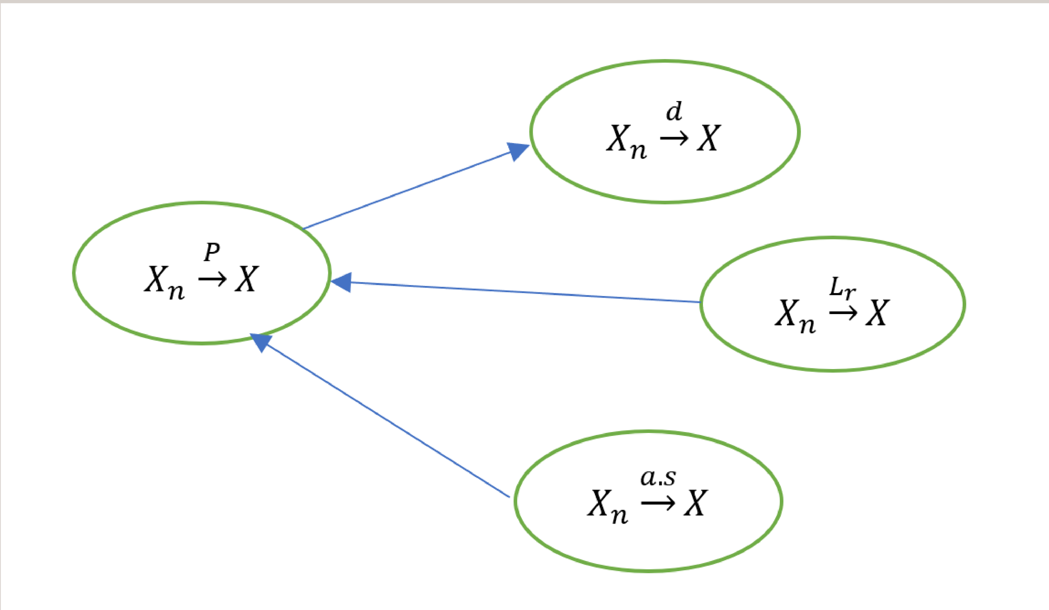

Figure 1. Relationship between various modes of convergence

Important Lemmas:

- If $X_n \xrightarrow{d} c \in \mathbb{R}$ where c is a constant, then $X_n \xrightarrow{P} c$.
- If $X_n \xrightarrow{a.s} X$, where $X_n \geq 0$ and $\mathbb{E}(X_n) \longrightarrow \mathbb{E}(X) < \infty$, then $X_n \xrightarrow{L_1} X$.
- $X_n \xrightarrow{P} X$ if and only if given any subsequence n_k , $\exists$ a further subsequence n_{k_l} such that $X_{n_{k_l}} \xrightarrow{a.s} X$ as $l \to \infty$.
- If $F_n \xrightarrow{d} F$ and F is continuous, then $\sup_{x \in \mathbb{R}} |F_n(x) - F(x)| \to 0$ as $n \to \infty$.

$\xrightarrow{P}, \xrightarrow{L^r}$ does not imply $\xrightarrow{a.s}$.

Let $\Omega = (0, 1)$ and let P corresponds to the $U(0, 1)$ distribution. Let

$$A_1 = (0, 1/2],\ A_2 = (1/2, 1),\ A_3 = (0, 1/4],$$
$$A_4 = (1/4, 1/2],\ A_5 = (1/2, 3/4],\ A_6 = (3/4, 1), ..$$

Let $X_n(\omega) = \mathbf{1}_{A_n}(\omega)$ for $n \geq 1$. Then $X_n \xrightarrow{P} 0$, and $X_n \xrightarrow{L_r} 0$ for any $r > 0$.
To see why, note that $P(|X_n| > \epsilon) = P(X_n = 1) = P(\omega \in A_n) \to 0$ as $n \to \infty$ (as the length of A_n converges to 0 as $n \to \infty$).
Similar argument holds for the L_r convergence. Next, fix any $\omega \in (0, 1)$. Observe that for any $\omega \in (0, 1)$, $X_n(\omega)$ takes both the values 0 and 1 infinitely often. Hence $X_n(\omega) \nrightarrow 0$ (in fact, $\lim_{n\to\infty} X_n(\omega)$ does not exist) for any $\omega \in (0, 1)$. Hence

$$P(\{\omega : X_n(\omega) \to 0 \text{ as } n \to \infty\}) = 0.$$

So, in particular, $X_n \overset{a.s}{\nrightarrow} 0$. Thus, convergence in probablity or convergence in L_r does not imply convergence almost surely.

$\xrightarrow{d}$ does not imply $\xrightarrow{P}$

$X_n \overset{i.i.d.}{\sim} N(0, 1)$. Then clearly $X_n \xrightarrow{d} X_1$. But does $X_n \xrightarrow{P} X_1$?
Why?: Fix any $\epsilon > 0$. Then for any $n \geq 2$, note that $P(|X_n - X_1| > \epsilon) = P(|V_n| > \epsilon)$ where $X_n - X_1 = V_n \sim N(0, 2)$. But clearly $P(|V_n| > \epsilon)$ is a non-zero positive quantity (independent of n) and can't converge to zero as $n \to \infty$. Thus $X_n \overset{P}{\nrightarrow} X_1$. Convergence in distribution does not imply convergence in probability.

Types of convergence

- **Convergence in Distribution**:
A sequence $\{X_n\}$ of random variables with d.f. F_n is said to converge in distribution or weakly (or, in law) to a random variable X with d.f. F if
$$\forall x \in \mathcal{C}_F,\ F_n(x) \to F(x).$$
We write $X_n \xrightarrow{d} X$ to denote that X_n converges to X in distribution.
- **Convergence in probability**: Let $\{X_n\}$ and X be defined as the same probablity space $(\Omega, \mathcal{A}, P)$. We say X_n converges in probablity to X, and write $X_n \xrightarrow{P} X$, if
$$\forall \epsilon > 0,\ P(|X_n - X| > \epsilon) \to 0 \text{ as } n \to \infty.$$
- **Almost sure convergence**: Let $\{X_n\}$, X be defined on the same probability space. We say X_n converges to X with probablity 1 (w.p. 1) or almost surely (a.s.) if
$$P(\{\omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)\}) = 1.$$
We write $X_n \xrightarrow{a.s} X$ or $X_n \xrightarrow{w.p.1} X$.
- **Convergence in r-th mean** Let $\{X_n\}$, X be defined on the same probability space. We say X_n converges to X in r-th mean if
$$\mathbb{E}(|X_n - X|^r) \to 0 \text{ as } n \to \infty.$$
We say $X_n \xrightarrow{L_r} X$. For $r = 2$, it is called convergence in quadratic mean.

Motivating example

Let $X_1, X_2, \ldots, X_n \overset{i.i.d.}{\sim} N(\mu, \sigma^2)$, σ^2 unknown. Then the usual *t-test* rejects $H_0 : \mu = \mu_0$ against $H_1 : \mu \neq \mu_0$ at significance level α if

$$|t| = \left| \frac{\sqrt{n}(\bar{x} - \mu_0)}{s} \right| > t_{\frac{\alpha}{2},(n-1)}$$

where symbols have their usual meanings. In the case of normal, this test has the verifiable optimality property that it is *Unbiased Most Powerful Uniformly*. However, it is a fact that t-test is very popular and is used all the times, normal data or not. Does Asymptotic theory give any justification for such universal use of the t-test? Yes, in a limited way. Suppose $X_1, X_2, \ldots, X_n \overset{i.i.d.}{\sim} F(x)$ and $\mathbb{E}_F(X_1) = \mu_0$. Then assuming that $\mathbb{E}_F({X_1}^2) < \infty$, you can show that

$$\mathbb{P}\left(|t| > t_{\frac{\alpha}{2},(n-1)}\right) \to \alpha \text{ as } n \to \infty.$$

Thus asymptotically the derived level α is preserved. So by asymptotic analysis, it becomes clear that the t-test is *level-robust* for any F with finite variance.

Results:

- If the in probability convergence occur at a rate fast enough so that
$$\forall \epsilon > 0,\ \sum_{n=1}^{\infty} \Pr(|X_n - X| > \epsilon) < \infty,$$
then we can say that $X_n \xrightarrow{a.s} X$.
- If $X_n \xrightarrow{P} X$, then there exists a subsequence n_k, such that $X_{n_k} \xrightarrow{a.s} X$ as $k \to \infty$.